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AFAPL-TR-65-45

Part II

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# ROTOR-BEARING DYNAMICS DESIGN TECHNOLOGY

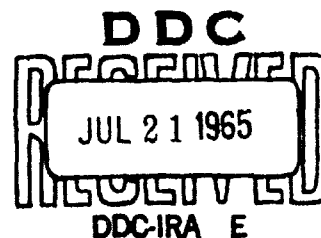
## Part II: Rotor Stability Theory

H. Poritsky

Mechanical Technology Incorporated

TECHNICAL REPORT AFAPL-TR-65-45, PART II

May 1965



Air Force Aero Propulsion Laboratory  
Research and Technology Division  
Air Force Systems Command  
Wright-Patterson Air Force Base, Ohio

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Part II: Rotor Stability Theory.

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# FOREWORD

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This technical report has been reviewed and is approved.

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ABSTRACT

This report treats the theory of the stability of an elastic rotor. It covers such effects as instability caused by friction damping in the rotor, instability caused by shaft asymmetry, instability caused by asymmetry of the rotor mass, instability induced by fluid film journal bearings, the effect of static damping on the stability of the rotor, the influence of gyroscopic effects, the effect of flexible bearing supports, and the effect of gravitational and unbalance forces.

The report gives the basic governing equations and the methods of their solution. The conditions are established under which instability is encountered and it is shown how the results relate to a practical rotor system.

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# I

## INTRODUCTION

In the following a review is given of stability of rotors for small displacements, both linear and angular, from the axis of rotation. Many of the results are well-known and are summarized here just for the sake of completeness. Some results are believed to be new, or at least were new when first obtained by the author.

The restriction to small displacements renders the dynamical equations of motion linear. This linearity allows one to use superposition, thus computing separately the rotor motion under effects of gravity, rotor unbalance, and the free motion of the rotor, that is, its motion near equilibrium, in absence of any external forces and torques. Most of the following is confined to a study of the free motion. The effects of external forces, however, are considered briefly in Section XI.

As regards the rotational speed of the rotor, it is assumed throughout most of the following that this speed is maintained constant. This may require a special drive, such as an electric motor, turbine, or belt drive. In order to simplify matters, the mechanism for insuring the constancy of speed is neglected, as is also the frictional torque. However, the effect of variable rotor speed is briefly considered in Section II, largely in running through the critical.

With gravity, unbalance or other external forces absent, the differential equations for the "free motion" of the rotor become both linear and homogenous. When the coefficients of these differential equations are constant, their solutions consist of linear combinations of exponentials

$$e^{\lambda_1 t}, e^{\lambda_2 t}, \dots \quad (1)$$

where the  $\lambda$ 's are roots of a proper algebraic equation, known as the "characteristic equation":

$$f(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0 \quad (2)$$

If all roots  $\lambda_1, \lambda_2, \dots, \lambda_n$  of (2) have a real negative part, then

every solution of the dynamical system representing displacements from equilibrium approaches zero as time increases. On the other hand, if any one of the roots of (2) has a positive real part, then the general solution increases indefinitely with time, and the equilibrium position is unstable.

From point of view of stability it is not really necessary to solve the characteristic algebraic equation (2) for all of its roots; it is sufficient to determine whether each root has a negative real part:

$$\operatorname{Re}(\lambda_i) < 0; \quad i = 1, 2, \dots, n. \quad (3)$$

In the following, such equations will be referred to, quite inaccurately, as "stable equations". Likewise, we refer to (2) as a "real equation", when its coefficients  $a_0, a_1, \dots, a_n$  are all real numbers, or can be made real by dividing (2) by a constant. An equation which is not "real" in the above sense, will be referred to as a "complex equation".

A well-known test for the stability of a real equation is due to Adolf Hurwitz (Ref.1). It is :

$$\text{Let} \quad a_0 > 0. \quad (4)$$

From the coefficient of (2) form the  $n \times n$  matrix

$$\begin{array}{ccccccc} a_1 & a_0 & 0 & 0 & \dots & & \\ a_3 & a_2 & a_1 & a_0 & \dots & & \\ a_5 & a_4 & a_3 & \dots & & & \\ \dots & \dots & \dots & \dots & & & \\ a_{2n-1} & \dots & & & & & a_n \end{array} \quad (5)$$

where the subscripts of the elements in the first column increase in steps of

2 from one row to the next one, while the subscripts of the elements in any one row decrease regularly; when the subscript is larger than that of any coefficient occurring in equation (2), or negative, that element is replaced by zero:

$$a_k = 0 \quad \text{for } k < 0 \quad \text{or } k > n. \quad (6)$$

A necessary and sufficient condition for the stability of the real equation (2) is that the following inequalities hold:

$$D_1 > 0, \quad D_2 > 0, \quad \dots \quad D_n > 0, \quad (7)$$

all hold, where  $D_1, \dots, D_n$  are the "principal" determinants of the matrix (5), given by

$$D_1 = a_1,$$

$$D_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix},$$

$$\dots$$

$$D_n = \begin{vmatrix} a_1 & a_0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{2n-1} & \dots & \dots & a_n & \dots \end{vmatrix}.$$

An alternative stability criterion for a real equation, due to Routh, (Ref. 2), will also be recalled.

Break up  $f(\lambda)$  into two parts as follows:

$$f(\lambda) = U(\lambda) + V(\lambda) \quad (9)$$

where

$$\begin{aligned} U(\lambda) &= a_0 \lambda^n + a_2 \lambda^{n-2} + \dots \\ V(\lambda) &= a_1 \lambda^{n-1} + \lambda^{n-3} + \dots \end{aligned} \quad (10)$$

where  $U$  contains all the terms of (2) with even subscripts (that is, the terms with  $a_0, a_2, \dots$ ), while  $V$  contains the terms with odd subscripts. Divide  $V$  into  $U$ , denoting the quotient by  $Q$  and the remainder by  $R$ :

$$V(\lambda) = U(\lambda) Q(\lambda) + R(\lambda). \quad (11)$$

Then, in order that the real equation (2) be stable it is necessary and sufficient that

$$a_0 a_1 > 0, \quad (12)$$

and (if  $n > 1$ ), the equation of degree  $(n-1)$ :

$$f_1(\lambda) = V(\lambda) + R(\lambda) \quad (13)$$

be stable. For  $n=1$  (12) is both necessary and sufficient.

Further applications of the above Routh criterion enable one to reduce the condition for stability to a number of inequalities of the form (12) and to the stability of an ever lower degree equation.

Turning to complex equations (2), two tests for stability will now be given.

A necessary and sufficient condition for the stability of a complex equation (2) is that the real equation of degree  $2n$ :

$$f(\lambda) \bar{f}(\lambda) = [a_0 \lambda^n + \dots + a_n] [\bar{a}_0 \lambda^n + \dots + \bar{a}_n] = 0 \quad (14)$$

be stable. Here bars denote conjugates:

$$\overline{c + id} = c - id \quad (15)$$

and  $\bar{f}(\lambda)$  is obtained from  $f(\lambda)$  by replacing all the coefficients of the latter by their conjugates.

A test for complex equations, which is similar to the Routh test given above for real equations, is as follows.

Let  $a_0$  in (2) be real, and denote the following complex coefficients thus:

$$a_k = b_k + i c_k, \quad c_0 = 0 \quad (16)$$

Break up  $f(\lambda)$  as in (9), where now

$$\begin{aligned} U(\lambda) &= b_0 \lambda^n + i c_1 \lambda^{n-1} + b_2 \lambda^{n-2} + i c_3 \lambda^{n-3} + \dots \\ V(\lambda) &= b_n \lambda^{n-1} + i c_2 \lambda^{n-2} + b_3 \lambda^{n-3} + \dots \end{aligned} \quad (17)$$

Divide  $V$  into  $U$ , as in (11), denote the quotient by  $Q$  and the remainder by  $R$ . Then, in order that the complex equation (2) be stable, it is necessary and sufficient that

$$b_0 b_1 < 0, \quad (18)$$

and that the complex equation (13), of degree  $n-1$ , be stable.

The last theorem was proved by the author in an internal General Electric report, "Stability, Hunting, Follow-Up", issued in 1938, where various other tests for stability were also presented. After a lapse of a somewhat long interval (26 years), it is planned to publish soon the rather interesting, purely geometric proof of it.

In certain cases the linearized rotor dynamical equations possess variable, periodic coefficients, generally of a period equal to half the period of rotation. It is known that special solutions of these equations exist, of the form

$$e^{\lambda_1 t} P_1(t) \quad (19)$$

where  $P_1$  are periodic, of the same period as the coefficients and  $\lambda_1$  are constant. The stability of the solutions now reduces to the requirement (3) for the exponential multipliers.

Unfortunately no simple test similar to the above criteria for stability of the solutions (19), of a differential equation with periodic coefficients exists. These equations will be briefly discussed in Section X.

# RADIAL ELASTIC AND STATIC FRICTION FORCES

Consider a rotor consisting of a disk of mass  $m$ , symmetrically supported on a relatively light shaft of stiffness  $k$ , and rotating with constant angular velocity about a fixed axis  $OO$  (see Fig.1). Assume no unbalance, neglect frictional torques, or

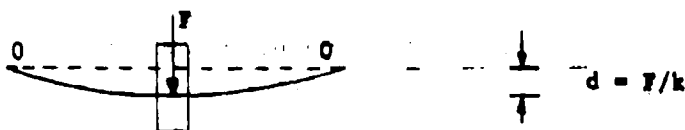


Figure 1

rather assume that any frictional torque is overcome by a proper driving torque, so that a constant speed of rotation is maintained.

Denote by  $x, y$  the displacement of the rotor (and shaft) center, relative to fixed (i.e. Newtonian)  $(x, y, z)$  - axes, with origin on the  $z$ -axis, at the center of the undeflected shaft. Then the equations of the rotor motion are

$$\ddot{mx} = -kx, \quad \ddot{my} = -ky. \quad (1)$$

In the above we have neglected forces other than the one due to the shaft stiffness  $k$ .

The solution of (1) leads to simply harmonic motion of  $x, y$ , of (radian) frequency

$$\omega_0 = \sqrt{k/m} \quad (2)$$

This will be called the "critical speed" of the rotor. More explicitly

$$x = A \cos(\omega_0 t + \epsilon), \quad y = B \cos(\omega_0 t + \epsilon') \quad (3)$$

If the phase angles  $\epsilon, \epsilon'$  are equal, then the path is a straight line through the origin. For general  $\epsilon, \epsilon'$ , the path may be shown to be an ellipse. If

$$\epsilon' = \epsilon \pm \pi/2, \quad A = \pm B \quad (4)$$

the path is a circle.

We next add to the shaft force a "frictional" force, opposing the velocity:

$$- f \dot{x}, - f \dot{y}, \quad (5)$$

where  $f$  is a positive constant. This force we term, somewhat inappropriately, "static" friction, in the sense that it opposes the velocity relative to static, i.e. fixed, axes. The equations of translational motion of the rotor center now become

$$\begin{aligned} m\ddot{x} &= -kx - f\dot{x}, \\ m\ddot{y} &= -ky - f\dot{y}. \end{aligned} \quad (6)$$

The solution of each one consists superposition of exponentials

$$e^{\lambda t}, \quad (7)$$

where  $\lambda$  satisfies the equation

$$m\lambda^2 + f\lambda + k = 0. \quad (8)$$

The two roots of (8) are given by

$$\lambda = \lambda_1, \lambda_2 = \frac{-f \pm \sqrt{f^2 - 4km}}{2m} \quad (9)$$

and each one has a negative real part. Thus the displacements from the undeflected position  $x = 0, y = 0$  approach zero with time; the effect of friction is to dampen out any initial disturbance from the equilibrium position, which position is thus seen to be a truly stable position.

In the above it will be noted that the rotation of the rotor has no effect on the bending of the shaft and the motion of the rotor center. In fact, Eqs. (1), (6) and (9) do not involve  $\omega$ , and they remain the same whether the rotor is rotating or not. The rotation of the rotor is merely superposed on the top of the displacement of the rotor center, the shaft rotating about its bent center line, the rotor and shaft center undergoing the displacement  $x, y$  corresponding to a force equal to the right-hand member of (6) applied at its center.

In general,  $f^2$  is small compared with the term  $km$ , and the roots  $\lambda$  in (9) may be put in the form

$$\lambda = -\frac{f}{2m} \pm i \sqrt{\frac{k}{m} - \left(\frac{f}{2m}\right)^2} = -\lambda_r \pm i\lambda_j, \quad (10)$$

where  $i^2 = -1$ ,  $-\lambda_r$  is the real part of  $\lambda$ , and  $\pm \lambda_j$  the imaginary part, leading to expressions for both  $x$  and  $y$  as linear combinations of products of

$$e^{-\lambda_r t}, \quad e^{\pm i\lambda_j t} \quad (11)$$

The pure imaginary exponentials represented by the second factor in (11) when properly combined, yield real sinusoidal terms of frequency  $\lambda_j$  given by

$$\lambda_j = \sqrt{\frac{k}{m} - \left(\frac{f}{2m}\right)^2} = \frac{k}{m} \left[ 1 - \frac{1}{2} \left(\frac{f^2}{4km}\right) - \frac{1}{8} \left(\frac{f^2}{4km}\right)^2 + \dots \right] \quad (12)$$

while the real exponential in (11), as stated above, leads to an exponential decay of amplitude. For  $f = 0$ , i.e. in absence of friction,  $\lambda_j$  reduces to the critical speed  $\omega_0$  given by (2), while for small  $f$ ,

$$\lambda_j = \omega_0 \left( 1 - \frac{f^2}{8km} \right). \quad (13)$$

The introduction of friction thus leads to a slight, second order decrease in the frequency  $\lambda_j$ . Nevertheless, we shall continue to refer to  $\omega_0$  (rather than  $\lambda_j$ ) as the critical speed of the rotor.

For  $f = 0$  the motion has been described above. For  $f > 0$ , the general motion may be viewed as describing an ellipse whose axes remain fixed in direction, but whose semi-axes are decreasing exponentially. For special motions the ellipses become circles and then the path is a logarithmic spiral; these are represented by the equations

$$x + iy = A e^{-\lambda_2 t} e^{i\lambda_j t}, \quad (14)$$

$$x + iy = B e^{-\lambda_2 t} e^{-i\lambda_j t}. \quad (15)$$

The general motion may also be viewed as a superposition of the two logarithmic spiral motions, given by Eqs. (14) and (15); also as obtainable from either (14) or (15) by effecting a homogeneous strain, or skewing, of the (x,y) plane, of the type in which parallel straight lines go into parallel straight lines, but distances and angles are not preserved.

Summarizing, it has been shown above that the effect of static friction on a balanced rotor consists in reducing any initial displacement of the rotor center to zero, both the x and the y coordinates varying with time as exponentially damped sinusoids. The motion of the rotor center takes place in an ellipse whose dimensions decrease exponentially with time, the frequency being given by  $\lambda_j$ , the damping being determined by  $\lambda_r$ , given by equation (10). Generally,  $\lambda_j$  is close to the critical frequency  $\omega_0$ , and  $\lambda_r$  is independent of the rotational speed of the rotor.

### III

#### RADIAL ELASTIC AND NORMAL FORCES

The elastic shaft force, for a round shaft, is directed along a radius, from  $x, y$  toward the center of the undeflected shaft; it may thus be described as a "radial" elastic (return) force. To this elastic force  $\tilde{F}_r$  in 2 (1) we now add a force  $\tilde{F}_n$ , normal to the displacement  $(x, y)$  and given by

$$\tilde{F}_n : (-k_1 y, k_1 x) \quad (1)$$

where  $k_1$  is a constant (see Fig. 2).

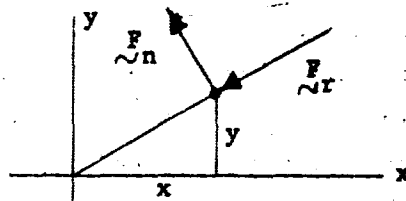


Figure 2

The equations of motion 2 (1) are replaced by

$$\begin{aligned} m\ddot{x} &= -kx - k_1 y, \\ m\ddot{y} &= -ky + k_1 x. \end{aligned} \quad (2)$$

In the system (2) the variables  $x, y$  do not separate as in 2 (1). This system can be solved by putting

$$x = X e^{\lambda t}, \quad y = Y e^{\lambda t} \quad (3)$$

leading to an algebraic equation in  $\lambda$ , of degree 4. A slight simplification results by proceeding as follows:

Multiply the second Eq. (2) by  $i = \sqrt{-1}$  and add to the first one. Introducing the variable

$$Z = x + i y \quad (4)$$

one is led to the second order differential equation

$$m\ddot{Z} = (-k + i k_1)Z \quad (5)$$

This is solved by putting

$$Z = A e^{\lambda t} \quad (6)$$

resulting

$$m\lambda^2 + (k - i k_1) = 0, \quad (7)$$

$$\lambda = \pm \sqrt{\frac{ik_1 - k}{m}} = \pm i \sqrt{\frac{k}{m} - \frac{ik_1}{m}}.$$

Since  $\lambda^2$  complex, one of its two roots, say  $\lambda_1$ , will have a real positive part, the other,  $-\lambda_1$ , a real negative part (see Fig.3).

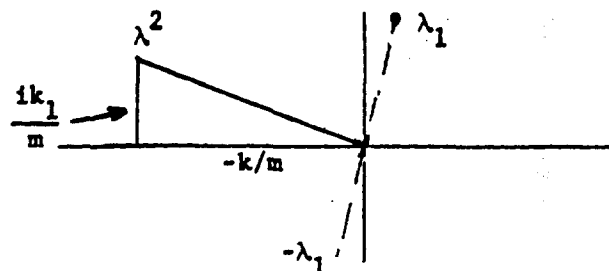


Figure 3

The general solution of (5) is

$$Z = A_1 e^{\lambda_1 t} + A_2 e^{-\lambda_1 t} \quad (8)$$

where  $A_1, A_2$  are arbitrary complex constants. The  $A_1$ -term (i.e. the term involving  $A_1$ ) becomes infinite with time; the  $A_2$ -term approaches zero. From (8)  $x$  and  $y$  are obtained by taking real and imaginary parts respectively.

Thus the effect of the force  $F_n$  normal to the displacement and proportional to it, is to make the rotor motion unstable. This conclusion holds whether  $k_1$  is positive, as in Figs. 2, 3, or negative, that is, whether the normal force is in the direction of rotation or in the opposite direction.

The path described by the  $A_1$  -term on the right of (8) represents motion in a logarithmic spiral, with  $|Z|$  increasing exponentially with time; the second term represents motion in a logarithmic spiral, of the same angular frequency but rotating in the opposite direction and  $|Z|$  decreasing exponentially with time. The general solution (8) is the vector resultant of the two motions, and as time gets large, approaches the receding logarithmic spiral.

The notion of a normal force is not purely an academic one. For a steam turbine or a gas turbine, bending of the rotor shaft decreases the radial clearance between the rotor and stator blades on the side of the displacement. This causes a decrease in the gas leakage over the edges of the rotor and stator blades, with a consequent increase in the local gas reaction on the rotor blades. On the opposite side an opposite effect takes place. However, the increase in force on the displacement side generally overbalances the decrease on the opposite side. Thus, there results a net (resultant) force  $\tilde{F}_n$ , normal to the displacement of the center.

It will be shown in Section IX, that the lubricant in a journal bearing, generally also produces a force  $\tilde{F}_n$  on the journal, normal to the displacement (in the bearing) of the journal center.

Returning to Eq(8) it is will be noted that the unstable term  $A_1 e^{\lambda_1 t}$  eventually leads to increasing energy of rotor motion. This energy can only grow from the work  $W$  done by the force  $\tilde{F}_r + \tilde{F}_n$ , during the rotor motion, where

$$\begin{aligned} W_{t_1}^{t_2} &= \int_{t_1}^{t_2} [(-kx - k_1 y) \dot{x} + (-ky + k_1 x) \dot{y}] dt \\ &= -k \left[ \frac{x^2 + y^2}{2} \right]_{t_1}^{t_2} + k_1 \int_{t_1}^{t_2} (-x\dot{y} + y\dot{x}) dt \end{aligned} \quad (9)$$

If  $k_1/k$  is small, then

$$\lambda_1 = i\omega_0 + \frac{1}{2} \frac{k_1}{k} \omega_0 \quad (10)$$

and the exponential growth of  $e^{\lambda_1 t}$  per cycle is small, and so is the decay of  $e^{-\lambda_1 t}$ . Then, for the  $e^{\lambda_1 t}$  term in (8), the integral in (9) per cycle becomes

$$k_1 r^2 \int d(\tan^{-1} \frac{y}{x}) = 2\pi k_1 r^2 = 2\pi k_1 |A_1|^2 \quad (11)$$

For the  $e^{-\lambda_1 t}$  it becomes

$$- 2 k_2 \pi |A_2|^2 ; \quad (12)$$

while for the cross terms the integrals vanish.

It is clear from the above that if  $|A_2|$  in (8) is much larger than  $|A_1|$ , then the rotor motion will be largely clockwise and the force  $F_n$  will take energy out of the rotor. Eventually the  $A_1$ -term will become equal to and exceed the  $A_2$ -term and then  $F_n$  will feed energy into the motion.

If, on the other hand, the direction of  $F_n$  were reversed in Fig. 4 by replacing  $k_1$  by  $-k_1$ , then  $F_n$  would cause decay of the  $e^{\lambda_1 t}$ -term and growth of the  $e^{-\lambda_1 t}$ -term.

This explains why the motion is unstable irrespective of the sign of  $k_1$ . Each sign promotes its own exponential motion.

#### IV

##### ROTARY FRICTION FORCE

We now return to the case considered in Section II, but replace the "static friction" force 2(5) by a "rotary friction" force. By this is meant a force proportional to, but opposite in direction of, the velocity of the rotor center relative to a rotating system of axes, rotating about the (fixed) z-axis with the same (constant) angular velocity  $\omega$  as the rotor. This relative velocity may be obtained by subtracting from the actual velocity  $(\dot{x}, \dot{y})$  the entrainment velocity due to the rotation of the system, given by  $(-\omega y, \omega x)$  (see Fig. 4). Denoting the coefficient of rotary friction by  $f_1$ , we now obtain for the equations of motion replacing 2(6).

$$\begin{aligned} m\ddot{x} &= -kx - f_1(\dot{x} + \omega y), \\ m\ddot{y} &= -ky - f_1(\dot{y} - \omega x), \end{aligned} \quad (1)$$

where  $\omega$  is the velocity of the rotor.

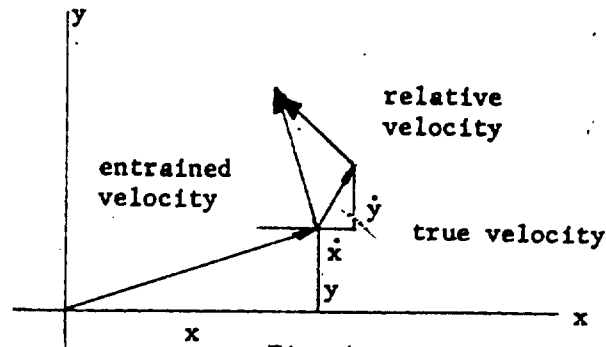


Fig. 4

Introducing  $Z$  as in 3(4), one obtains

$$m\ddot{Z} + f_1\dot{Z} + (k - if_1\omega)Z = 0. \quad (2)$$

Once more this can be solved as in 3(6) where  $\lambda$  is now the root of the complex equation

$$m\lambda^2 + f_1\lambda + (k - if_1\omega) = 0. \quad (3)$$

To insure stability each root  $\lambda$  must have a real negative part.

For a second degree complex algebraic equation

$$b_0 \lambda^2 + (b_1 + ic_1)\lambda + (b_2 + ic_2) = 0 \quad (4)$$

where  $b_0, b_1, \dots, c_2$  are real, the test described in the text accompanying Eqs. 1(16), 1(17) leads to the following inequalities,

$$\begin{aligned} b_0 b_1 &> 0, \\ (b_1 b_2 + c_1 c_2) - \frac{b_0 c_2^2}{b_1} &> 0, \end{aligned} \quad (5)$$

as a necessary and sufficient condition for the stability of (4).

Applying (5) to (3) one is led to

$$\begin{aligned} mf_1 &> 0, \\ f_1 (k - m\omega^2) &> 0. \end{aligned} \quad (6)$$

The first inequality in (6) is automatically satisfied (since  $f_1$  is positive). The second one leads to

$$\omega < k/m = \omega_0. \quad (7)$$

Thus, for rotor speeds below the critical, the rotor motion is stable.

For rotor speeds greater than the critical -

$$\omega > \omega_0 \quad (8)$$

the second inequality in (6) will fail, at least one of the roots of (3), say  $\lambda_1$ , will have a real positive part, and the rotor motion will be unstable. Since the sum of the two roots is

$$\lambda_1 + \lambda_2 = -\frac{f_1}{m} \quad (9)$$

it follows that for

$$\operatorname{Re}(\lambda_1) > 0, \quad \operatorname{Re}(\lambda_2) < 0. \quad (10)$$

The transition to instability occurs for

$$\omega = \omega_0 = \sqrt{\frac{k}{m}} \quad (11)$$

when  $\lambda_1$  is pure imaginary. Eq. (3) now yields

$$\begin{aligned} m \lambda_1^2 + k &= 0, \\ f_1 \lambda_1 - i f_1 \omega_0 &, \end{aligned} \quad (12)$$

and hence

$$\lambda_1 = i \omega_0. \quad (13)$$

Thus the transition to instability occurs when the rotor is running at the critical speed, and it is characterized by a circular synchronous whirl in the direction of rotation, with the shaft "frozen" in its whirling, deflected shape.

It follows from (9) that the root  $\lambda_2$  leads to an oppositely rotating whirl, which dampens out with time.

In the unstable range (8), the root  $\lambda_1$  yields an exponentially increasing whirl. While the unstable range is independent of  $f_1$ , it can be shown that the rate of growth of instability increases with  $f_1$ .

The notion of "rotary" friction is, itself, by no means academic. Rotary friction is produced by any device, such as rubber pads or rubbing surfaces mounted on the rotor. Shrink fits, especially of wide disks or plates shrunk on the shaft, are another cause of rotary friction, since if the shaft bends the shrunk element does not follow the axial extension and compression due to bending of the shaft. Thus, slippage takes place along the shrunk surfaces when the shaft bending changes, and the shaft center displaces relative to a system of axes rotating about the z-axis with the rotor speed.

Likewise "solid friction" of the shaft material contributes to the rotary friction; by "solid friction" is implied the small hysteresis loop obtained by plotting the strain vs stress for a periodic stress-strain: even in the elastic range, the curve is not a straight line but forms a thin loop whose area is a measure of the energy loss per unit volume, per cycle. For stresses within the elastic limit, the energy loss per cycle is of the form

$$f V \quad (14)$$

where  $V$  is the maximum elastic strain energy and  $f$  is a "solid friction" coefficient, characteristic of the shaft material. For larger stresses  $f$  is not constant, but increases with amplitude.

It is not evident that the above rotary friction effects are truly represented by the  $f_1$ -terms in (1), namely as a "viscous" force, proportional to, but opposing the relative velocity. The following considerations show in fact that they are not.

For sinusoidal strains and displacements, solid friction effects are more correctly treated by replacing the Young's modulus  $E$  by the complex number

$$(1 - i s) E, \quad i = \sqrt{-1} \quad (15)$$

Thus, for the complex solution 3(4), if time is assumed to enter as a factor

$$e^{ipt}, \quad (16)$$

Eq. 3(2) can be treated by replacing the shaft stiffness constant by

$$k(1 - i s) \quad (17)$$

and this is analogous to Eq. 2 (6) if the friction terms  $- \dot{f}_x$ ,  $- \dot{f}_y$  are replaced by  $- k_1 x$ ,  $- k_1 y$  corresponding to a value of  $s$  in (15) proportional to the frequency:

$$s = k p. \quad (18)$$

It is of interest to note from Eqs. (1), (2) that the "rotary friction" force terms involving  $f_1$  can be represented as the sum of

$$- f_1 \dot{Z} + i f_1 \omega Z \quad (19)$$

of a static damping force and a force perpendicular to the displacement, and analogous to  $F_n$  in 3(1) but with a constant  $k_1$  proportional to the rotor speed:

$$k_1 = f_1 \omega. \quad (20)$$

# V STATIC AND ROTARY FRICTION, NORMAL FORCE

We now consider a rotor possessing both static and rotary friction in a manner similar to that of Sections II & IV. The equations of motion are

$$m\ddot{x} + f\dot{x} + f_1(\dot{x} + \omega y) + kx = 0, \quad (1)$$

$$m\ddot{y} + f\dot{y} + f_1(\dot{y} - \omega x) + ky = 0.$$

Introducing  $Z$  as in 3(4), one obtains

$$m\ddot{Z} + (f + f_1)\dot{Z} + (k - if_1\omega)Z = 0. \quad (2)$$

One is led to solutions 3(6) where  $\lambda$  are roots of

$$m\lambda^2 + (f + f_1)\lambda + (k - if_1\omega) = 0. \quad (3)$$

Applying the criteria 4(5) for stability of this complex equation, one obtains the inequalities:

$$m(f + f_1) > 0, \quad (4)$$

$$(f + f_1)^2 k - mf_1^2 \omega^2 > 0.$$

The first inequality is again automatically satisfied, while the second one can be put in the dimensionless form

$$\left(1 + \frac{f}{f_1}\right)^2 > \left(\frac{\omega}{\omega_0}\right)^2 \rightarrow \left(\frac{\omega}{\omega_0}\right)^2 \quad (5)$$

leading to

$$\frac{f}{f_1} > \frac{\omega}{\omega_0} - 1. \quad (6)$$

Two cases arise. If the rotational speed is less than the critical speed  $\omega_0$ , then the right-hand member of (6) is negative, and (6) is automatically satisfied irrespective of what the values of  $f$ ,  $f_1$  are. On the other hand, if  $\omega$  is greater than  $\omega_0$ , the right-hand member of (6) is positive, and (6) will be satisfied provided that the static friction coefficient  $f$  is sufficiently large compared with the rotary coefficient  $f_1$ .

The relation between  $\omega/\omega_0$  and  $f/f_1$  is shown in Fig. 5 where the solid line AB marks the boundary between the region of stability and instability. We thus arrive at the criterion that if the rotor is running above its critical speed, there will be stability provided that the static friction be sufficiently large

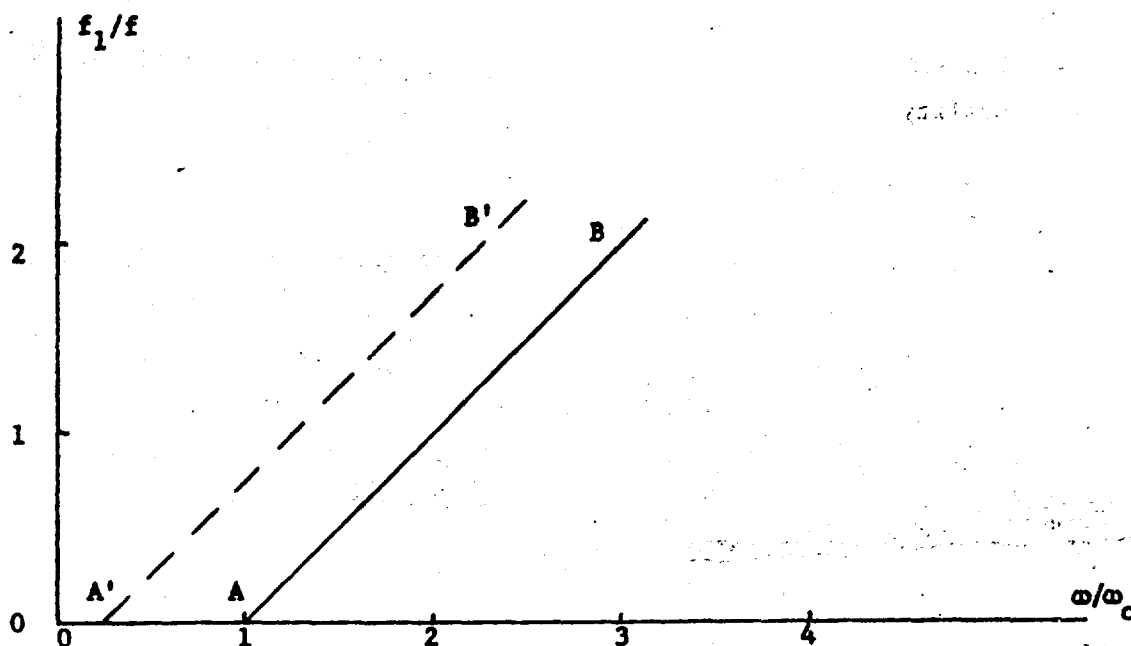


Fig. 5

compared to the rotary friction or, what amounts to the same thing, that the rotary friction be sufficiently small compared to the static friction.

As an example, for  $\omega/\omega_0 = 2$  corresponding to a rotor speed which is double the critical speed,  $f/f_1$  must exceed unity, that is, the coefficient of static friction  $f$  must at least be equal to the coefficient of rotary friction  $f_1$ , to insure stability. For  $\omega/\omega_0 = 2$ , if  $f_1 > f$ , there will be instability; if  $f_1 < f$ , stability will obtain.

In the unstable range the rate of increase of whirl increases with  $f_1/f$ .

Finally, if to the forces in (1) one also adds the normal force 3(1), one is led to

$$\begin{aligned} m\ddot{x} &= -kx - k_1 y - f\dot{x} - f_1(\dot{x} + \omega y), \\ m\ddot{y} &= -ky + k_1 x - f\dot{y} - f_1(\dot{y} - \omega x). \end{aligned} \quad (7)$$

Introducing  $Z$  as in III (4), one is led to

$$m\ddot{Z} + (f + f_1) \dot{Z} + (k - i k_1 - i f_1 \omega) Z = 0 \quad (8)$$

and to exponential solutions for  $Z$  as in 3(6) where

$$m\lambda^2 + (f + f_1)\lambda + k - i(k_1 + \omega f_1) = 0. \quad (9)$$

By applying similar techniques, one now obtains for stability of (9) the following inequality

$$\frac{f_1}{f_1} > \frac{\omega}{\omega_0} - 1 + \frac{k_1 \omega_0}{f_1} \quad (10)$$

The boundary of the region of stability is shown as the dotted slanting line A' B' on Fig. 5, for positive  $k_1$ ; for negative  $k_1$ , A'B' is to the right of AB.

Remarks similar to those made at the ends of Secs III, IV apply equally well here. In some cases it is possible to fit the above theory so as to include effects of solid friction and other aspects of frictional losses, by assuming that the "constants"  $k_1$ ,  $f$ ,  $f_1$ , vary with  $\omega$  (or even with the amplitude of oscillation). When this is done, it is evident that the rectilinear boundaries of the stability region of Fig. 5 may be greatly distorted.

## VI

### DIFFERENT SHAFT STIFFNESS CONSTANTS

We now consider a rotor mounted on a shaft of uniform but non-circular cross-section. A normal section of such a shaft possesses two mutually perpendicular directions corresponding to the maximum and minimum moments of inertia and it possesses different bending stiffness constants  $k_1, k_2, k_2 > k_1$ , in these two mutually perpendicular directions. One may define two vibrations frequencies

$$\omega_1 = \sqrt{\frac{k_1}{m}}, \quad \omega_2 = \sqrt{\frac{k_2}{m}}, \quad \omega_2 > \omega_1. \quad (1)$$

As a result of the two stiffnesses it can be shown that a force in a direction different from the principal directions of inertia produces shaft deflections lying in a plane different from the plane through the z-axis containing the force. To find the displacement one resolves the force along the principal directions of the shaft section, determines the deflection along each one and superposes them.

If the shaft is rotating and the force is fixed, then the deflection components turn out to involve in the coefficients  $\sin 2\omega t, \cos 2\omega t$ , where  $\omega$  is the velocity of rotation of the shaft.

As a result of the above the differential equation of motion also involves the above trigonometric functions.

To avoid this complication we refer the motion to a system of axes,  $(x, y)$  which rotates about the fixed z-axis with the rotor (and shaft) speed  $\omega$ . The deflection of the rotor center is given by

$$\tilde{x}i + \tilde{y}j \quad (2)$$

where  $\tilde{i}, \tilde{j}$  are unit vectors along the (rotating) x, y-axes.

These  $\tilde{i}, \tilde{j}$  are not constant, but

$$\frac{d\tilde{i}}{dt} = \omega \tilde{j}, \quad \frac{d\tilde{j}}{dt} = -\omega \tilde{i}. \quad (3)$$

Differentiating (2) we have for the (true) velocity

$$\underline{\dot{v}} = \dot{x} \underline{i} + \dot{y} \underline{j} + x \frac{d\underline{i}}{dt} + y \frac{d\underline{j}}{dt}, \quad (4)$$

and substituting from (3)

$$\underline{v} = (\dot{x} - \omega y) \underline{i} + (\dot{y} + \omega x) \underline{j} \quad (5)$$

Further differentiation yields for the acceleration

$$\underline{a} = (\ddot{x} - 2\omega\dot{y} - \omega^2 x) \underline{i} + (\ddot{y} + 2\omega\dot{x} - \omega^2 y) \underline{j}. \quad (6)$$

Hence the dynamical equations of the rotor are

$$\begin{aligned} m(\ddot{x} - 2\omega\dot{y} - \omega^2 x) &= -k_1 x, \\ m(\ddot{y} + 2\omega\dot{x} - \omega^2 y) &= k_2 y. \end{aligned} \quad (7)$$

This system has constant coefficients. The variables  $x, y$  in (7) do not separate. Nor can these equations be simplified by introducing complex variables as in 3(4).

Assuming a solution of the form 3(3) one is now led to the fourth degree equation

$$(\lambda^2 - \omega^2 + \omega_1^2)(\lambda^2 - \omega^2 + \omega_2^2) + 4\omega^2 \lambda^2 = 0. \quad (8)$$

This is quadratic in  $\lambda^2$ . Hence, unless both roots  $\lambda^2$  are negative, the solution Eq. (7) will be unstable.

Put (8) in the form

$$\lambda^4 + 2\lambda^2 b + c = 0, \quad (9)$$

where

$$\begin{aligned} b &= (\omega_1^2 + \omega_2^2) / 2 + \omega^2 > 0 \\ c &= (\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2). \end{aligned} \quad (10)$$

The  $\lambda^2$  roots of (9) are

$$\lambda^2 = -b \pm \sqrt{b^2 - c} \quad (11)$$

The radicand above is

$$b^2 - c = \left( \frac{\omega_1^2 + \omega_2^2}{2} \right)^2 + 2\omega^2 (\omega_1^2 + \omega_2^2) > 0, \quad (12)$$

ans is always positive. Hence the roots (11) are always real. If  $c$  is positive, then both roots (11) are negative. On the other hand, if  $c$  is negative, then the root  $-b + \sqrt{b^2 - c}$  will be positive, and some solutions of (7) will be unstable. Examination of (10) shows that this happens for

$$\omega_1^2 < \omega^2 < \omega_2^2 \quad (13)$$

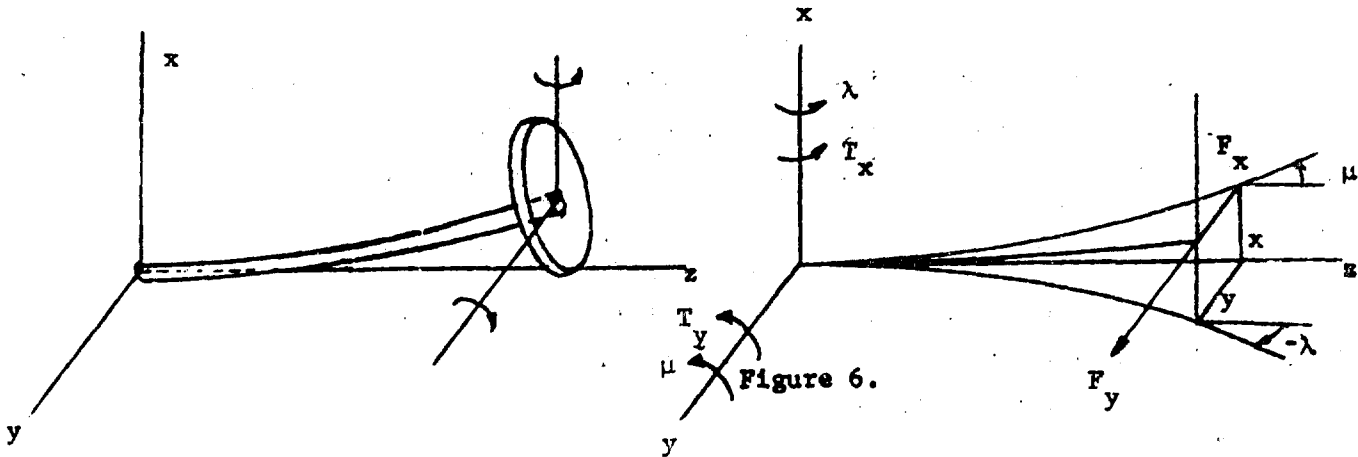
Thus, for a rotor with a shaft possessing two different stiffness constants in two mutually perpendicular directions, the speed range (13) between the two vibration frequencies (1) is unstable.

If static friction also exists, then the range of instability narrows, and for a sufficiently large value of the coefficient  $f$ , may disappear altogether. On the other hand, normal forces and rotary friction may enlarge it.

Thus far we have considered only translational displacements of the rotor. Angular displacements of the rotor axis, it will be shown in Section VIII, introduce gyroscopic effects, and may produce a further unstable range at high enough speeds.

GYROSCOPIC EFFECTS

Even when the rotor is mounted as in Fig. 1, it may execute both translational motion, normal to the z-axis, as well as rotational motions about transverse axes. The latter motions are even more likely to occur when the rotor is mounted asymmetrically, for instance if it is overhung, as indicated schematically in Fig. 6. The following analysis applies to any mode of support of the rotor.



Let the z-axis contain the undeflected shaft center line position, and denote by  $x, y$  the normal translational displacements of the rotor center, and by  $\lambda, \mu$  its angular (or rotational) displacements about the x, y-axes; all four quantities  $x, y, \lambda, \mu$  are assumed to be small.

In this Section we assume that the shaft is axially symmetric elastically, and that the rotor is axially symmetric; in Section VIII we consider the case when the shaft stiffness constants in two mutually perpendicular directions are different, and the rotor has different moments of inertia  $I, J, I \neq J$ , about the principal axes of inertia, normal to the spin axis; the third principal moment of inertia, about the spin axis, will be denoted by  $K$ .

For the axially symmetric shaft (and rotor), a force  $F_x$ , applied to the shaft at the rotor center, and a moment  $T_y$  (about the y-axis), produce a linear displacement  $x$  and a rotation  $\mu$  about the y-axis, given by

$$\begin{aligned} x &= a F_x + b T_y, \\ \mu &= b F_x + c T_y, \end{aligned} \quad (1)$$

where  $a, b$  are the linear and angular displacements produced by a unit force  $F_x$

acting alone;  $b, c$  the similar displacements produced by a unit torque  $T_y$  about the  $y$ -axis, acting alone. The matrix of coefficients  $a, b, c$  in (1) is the "flexibility matrix". Solution of (1) yields

$$\begin{aligned} F_x &= Ax + B\mu, \\ T_y &= Bx + C\mu, \end{aligned} \quad (2)$$

where the matrix  $A, B, C$  is the "stiffness matrix", and is the inverse of the matrix in (1).

Similarly, an applied force  $F_y$  and moment  $T_x$  (about the  $x$ -axis) produce displacements in the  $(y, z)$ -plane, given by

$$\begin{aligned} y &= a F_y - b T_x, \\ \lambda &= -b F_y + c T_x, \end{aligned} \quad (3)$$

and by the inverse equations

$$\begin{aligned} F_y &= Ay - B\lambda, \\ T_x &= -By + C\lambda. \end{aligned} \quad (4)$$

If the  $z$ -axis is a stable position of equilibrium, then the quadratic form

$$Q = (Ax^2 + 2 Bx\mu + C\mu^2) / 2 \quad (5)$$

representing the elastic energy corresponding to the deflection  $(x, \mu)$ , is positive definite. The conditions for this are

$$A > 0, AC - B^2 > 0, \quad (C > 0) \quad (6)$$

(where  $C > 0$  follows from the preceding conditions). Similar remarks apply to the matrices (1), (3), (4).

It is assumed above that the displacements, both linear and angular, are small, so that superposition applies, and the order of the rotations  $\lambda, \mu$  is immaterial.

It is further assumed that the rotor is balanced about its axis of rotation; this assumption implies that

- (a) the center of mass (c.m.) of the rotor lies on its axis of rotation;
- (b) the axis of rotation is a principal axis of inertia of the rotor, through its c.m.

These conditions are assumed to obtain. The assumed rotational symmetry implies equality of the two transverse moments of inertia,  $I = J$ . Hence any two axes through the c.m. mutually perpendicular and normal to the  $z$ -axis, can be considered as principal axes of inertia of the rotor in its undeflected position. In particular the unit vectors  $\hat{i}, \hat{j}, \hat{k}$ , parallel to the  $x, y, z$ -axes,

can be used as principal directions of inertia, even though they are fixed in space and hence not in the rotor. After the (small) angular deflections  $\lambda, \mu$ , the above principal directions vectors become:

$$\begin{aligned}\tilde{i}_1 &= \tilde{i} - \tilde{k}\mu, \\ \tilde{j}_1 &= \tilde{j} + \tilde{k}\lambda, \\ \tilde{k}_1 &= \tilde{i}\mu + \tilde{j}\lambda + \tilde{k}.\end{aligned}\quad (7)$$

Now let  $\lambda, \mu$  vary with time. The motion of the rotor R, at any time instant, consists of a translational velocity of its c.m.

$$\underline{v} = \dot{\tilde{x}}\tilde{i} + \dot{\tilde{y}}\tilde{j}, \quad (8)$$

and of a rotation represented by a proper rotation vector through its c.m. given by

$$\underline{\omega} = \dot{\lambda}\tilde{i} + \dot{\mu}\tilde{j} + \omega\tilde{k}_1 \quad (9)$$

assuming that R continues to rotate about its bent shaft with velocity  $\omega$ .

Utilizing Eq.(7),  $\omega$  can also be put in the form

$$\underline{\omega} = \dot{\lambda}\tilde{i}_1 + \dot{\mu}\tilde{j}_1 + \omega\tilde{k}_1 \quad (10)$$

provided second order terms in  $\lambda, \mu$ :  $\dot{\lambda}, \dot{\mu}$  are neglected. This form yields the component of  $\omega$  along the (instantaneous) principal axes, and this simplifies the expression of  $\underline{M}$ , the moment of the momentum, to be referred as m.m.:

$$\begin{aligned}\underline{M} &= I(\dot{\lambda}\tilde{i}_1 + \dot{\mu}\tilde{j}_1) + K\omega\tilde{k}_1 \\ &= I(\dot{\lambda}\tilde{i} + \dot{\mu}\tilde{j}) + K\omega(\tilde{i}\mu - \tilde{j}\lambda + \tilde{k}),\end{aligned}\quad (11)$$

where the last form is obtained again by using (7) and neglecting second order terms.

Differentiation of  $\underline{M}$  yields

$$\dot{\underline{M}} = (I\ddot{\lambda} + K\dot{\mu}\omega)\tilde{i} + (I\ddot{\mu} - K\dot{\lambda}\omega)\tilde{j}. \quad (12)$$

We now proceed to set up the dynamical equations of motion of R. Recall that in Eq. (1)  $F_{x\tilde{i}} + F_{y\tilde{j}}$  is the force applied to the shaft at the c.m. of R; this force is either an external force or it is applied by R to the shaft.

The reaction force,  $-F_{xj} - F_{yj}$ , is therefore applied by the shaft to R. Similarly the shaft applies the torque  $-T_{xi} - T_{yj}$  to R. Equating these shaft reactions on R to the  $m\ddot{x}$  and  $M\ddot{\lambda}$ , we obtain the dynamical equations

$$\begin{aligned} m\ddot{x} + Ax + B\mu &= 0, \\ m\ddot{y} + Ay - B\lambda &= 0, \\ I\ddot{\lambda} + K\omega - By + C\lambda &= 0, \\ I\ddot{\mu} - K\omega + Bx + C\mu &= 0. \end{aligned} \quad (13)$$

Assuming a solution of (13) in which time enters as a factor

$$e^{vt}, \quad (14)$$

and rearranging the order of the equations there results the following determinantal equation

$$\Delta(v) = \begin{vmatrix} \overset{x}{A + m v^2} & \overset{\mu}{B} & \overset{y}{0} & \overset{\lambda}{0} \\ B & C + I v^2 & 0 & -K \omega v \\ 0 & 0 & A + m v^2 & -B \\ 0 & K \omega v & -B & C + I v^2 \end{vmatrix} = 0, \quad (15)$$

where the coefficients in any one column arise from the variable indicated in that column, in the row above the determinant.

The function  $\Delta(v)$  is evidently even in  $v$ , and of the fourth degree in  $v^2$ . Thus, unless all four roots  $v^2$  are negative, at least one root will have a real positive part and the solution of (13) will be unstable.

We expand  $\Delta(v)$  in a Laplace expansion in terms of its first two columns:

$$\begin{aligned} \Delta &= (12) \begin{bmatrix} 34 \end{bmatrix} - (13) \begin{bmatrix} 24 \end{bmatrix} + (14) \begin{bmatrix} 23 \end{bmatrix} \\ &+ (23) \begin{bmatrix} 14 \end{bmatrix} - (24) \begin{bmatrix} 13 \end{bmatrix} + (34) \begin{bmatrix} 12 \end{bmatrix} \end{aligned} \quad (16)$$

where  $(ij)$  is the 2nd order determinant in the first two columns and rows  $i, j$  while  $\begin{bmatrix} ij \end{bmatrix}$  is the determinant in the last two columns and the same rows.

There results

$$\begin{aligned} \Delta &= (12)^2 + (14)^2 \\ &= \left[ (A + m v^2) (C + I v^2) - B^2 \right]^2 + \left[ (A + m v^2) K v \omega \right]^2 = 0 \end{aligned} \quad (17)$$

To consider negative  $v^2$ , we put

$$v = ip, \quad v^2 = -p^2. \quad (18)$$

Then  $\Delta$  can be factored thus

$$\Delta = \left\{ \begin{aligned} &[(A - mp^2)(C - Ip^2) - B^2] - (A - mp^2) K p \omega \\ &[(A - mp^2)(C - Ip^2) - B^2] + (A - mp^2) K p \omega \end{aligned} \right\}. \quad (19)$$

The first factor is linear in  $\omega$  and, when equated to zero, yields

$$\omega = f(p) = \frac{1}{K p} \left[ (C - Ip^2) - \frac{B^2}{A - mp^2} \right] \quad (20)$$

The right hand member is odd in  $p$ , and has the asymptotes

$$p = 0, \quad p = \pm \sqrt{A/m}, \quad \omega = -(I/K) p. \quad (21)$$

Near  $p = 0$

$$\omega = \frac{(CA - B^2)}{KA} \frac{1}{p} + \dots \quad (22)$$

where  $\dots$  is analytic at  $p = 0$ . If we assume that the  $x$ -axis is a position of stable equilibrium, then [noting Eq. (6)] the residue at  $p = 0$  is positive. Likewise the residue at  $p = \sqrt{A/m}$  is positive.

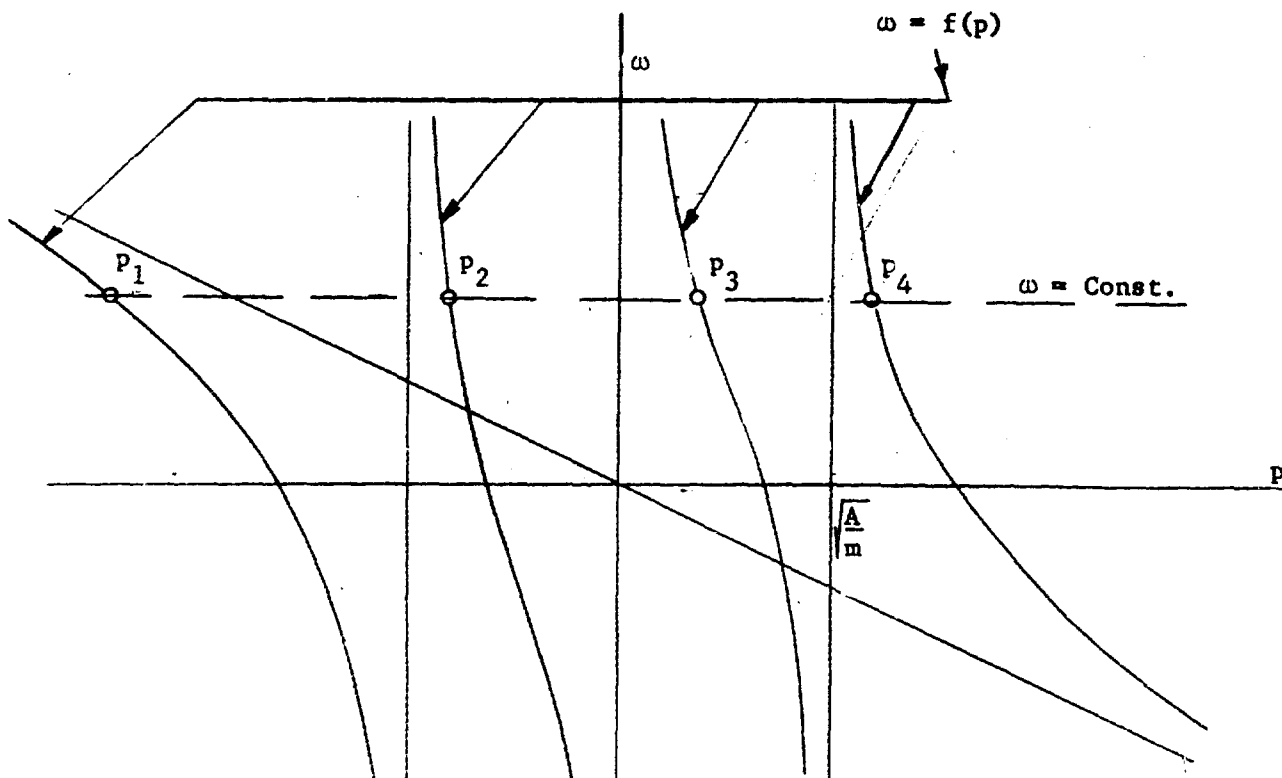


Figure 7

A schematic plot of Eq. (20) is shown in Fig. 7. It is evident that a line of constant  $\omega > 0$ , cuts this curve in 4 distinct points.

$$p = p_1, p_2, p_3, p_4, \quad (23)$$

two positive, two negative; but these roots cannot be negatives of each other. They correspond to the time factors.

$$e^{ipt}, \quad p = p_j, \quad j = 1, 2, 3, 4. \quad (24)$$

The second factor in (19) leads to

$$\omega = -f(p) \quad (25)$$

and hence on Fig. 7 to the reflection of the curve shown in the p-axis, and to 4 roots which are the negatives of (23).

It has thus been shown that all four roots  $\sqrt{\Delta}$  of  $\Delta$  are indeed negative and distinct. Thus for a symmetric rotor and shaft the gyroscopic effects never lead to instability.

The factorization of  $\Delta$  suggests that the system (13) may be reducible to two systems of lower order.

By multiplying the second Eq. (13) by  $i$  and adding to the first one, the third one by  $-i$  and adding to the fourth one, one is led to the second order system of differential equations

$$\begin{aligned} m \ddot{Z} + A \dot{Z} + B M &= 0, \\ B \dot{Z} + I \ddot{M} - i K \omega M + C M &= 0, \end{aligned} \quad (26)$$

where

$$Z = x + iy, \quad M = \mu - i\lambda \quad (27)$$

A conjugate set of equations can be derived for the variables

$$\bar{Z} = x - iy, \quad \bar{M} = \mu + i\lambda. \quad (28)$$

Eqs. (26) and their conjugate equations could have been used in place of Eqs. (13).

Indeed, assuming a solution of (26) in which time enters as a factor  $e^{ipt}$  yields the frequency equation

$$\begin{vmatrix} (A - mp^2) & B \\ B & (C + K\omega p - Ip^2) \end{vmatrix} = 0 \quad (29)$$

and hence leads to Eq. (20).

Each of the four roots of (23) results in a solution of the form

$$x + iy = Z_0 e^{ipt}, \quad \mu - i\lambda = M_0 e^{ipt}. \quad (30)$$

Taking real parts one may interpret the motion as a whirl of the rotor, in which the shaft executes a rotation of frequency  $p$ , with each point on the axis describing a circle and the axis as a whole a hyperboloid of rotation. This motion may be described as a circular whirl. In particular the point on the rotor axis at a distance  $z$  from the origin describes the circular path

$$x + iy + (z - z_0)(\mu - i\lambda) = [Z_0 + (z - z_0)M_0] e^{ipt} \quad (31)$$

where  $z = z_0$  corresponds to the rotor c

The first factor in (19), put in the form

$$(A - mp^2) \left[ C - (I - K \frac{\omega}{p}) p^2 \right] - B^2 = 0, \quad (32)$$

shows that for each circular whirl Eq. (32) is the same as the equation of vibration of a non-rotating rotor whose transverse moment of inertia  $I$  is replaced by

$$I' = I - K(\omega/p) \quad (33)$$

In certain cases this may lead to negative  $I'$ , but always to real frequencies of vibration.

Similarly, from the conjugate equations for  $\bar{Z}$ ,  $\bar{M}$ , one obtains four circular whirls corresponding to the negatives of the roots of (29).

As a special case, suppose that the shaft is rigid, and is restrained toward  $z$ -axis by springs of (angular) stiffness  $C$ . Then the equations of motion corresponding to the moment of momentum equations about the origin suffice for describing the dynamical behavior, and they lead to

$$\begin{aligned} I\ddot{\lambda} + K\dot{\mu}\omega + C\lambda &= 0, \\ I\ddot{\mu} - K\dot{\lambda}\omega + C\mu &= 0, \end{aligned} \quad (34)$$

where now  $I$  is the moment of inertia of the rotor about the  $x$ - and  $y$ -axes.

One is led to the frequency equation

$$(I p^2 - C)^2 - K p^2 \omega^2 = 0. \quad (35)$$

Solving for  $\omega$ , there results

$$\omega = \pm (I p^2 - C) / K p. \quad (36)$$

A plot of  $\omega$  vs  $p > 0$  is shown in Fig. 8. Again, for each constant  $\omega$ , there are now two real, distinct values of  $p$ , corresponding to two circular precessions of whirls, both positive; there are also two negative ones (not shown).

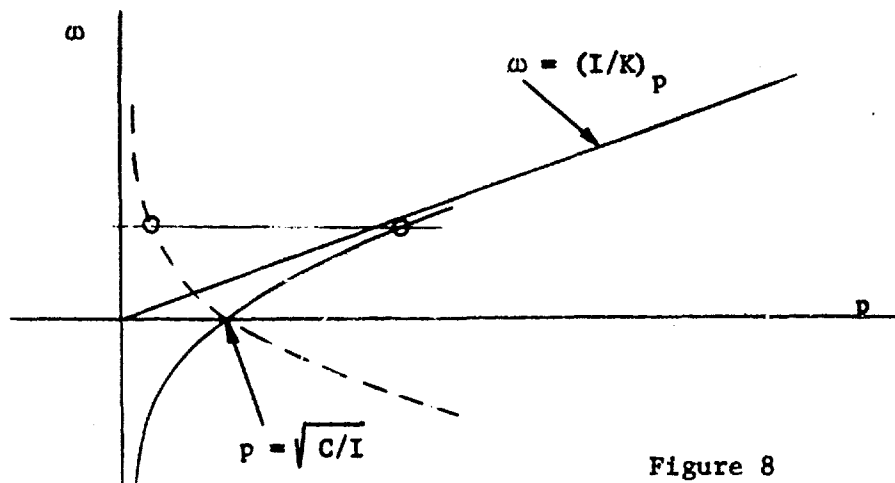


Figure 8

It is of interest to note that if the  $z$ -axis is not a stable equilibrium position, then the residue in (22) is negative, and some of the real roots of (20) may be lost, leading to positive  $v^2$  and unstable solutions, especially for small  $\omega$ .

For the special case implied in Eqs. (34) such an unstable equilibrium along the  $z$ -axis corresponds to negative  $C$ . For negative  $C$ , Fig. 8 is modified into Fig. 9, with a minimum for  $\omega$  at the point

$$p = p_0 = \sqrt{\frac{-C}{I}}, \quad \omega = \omega_0 = \frac{2I}{K} p_0 \quad (37)$$

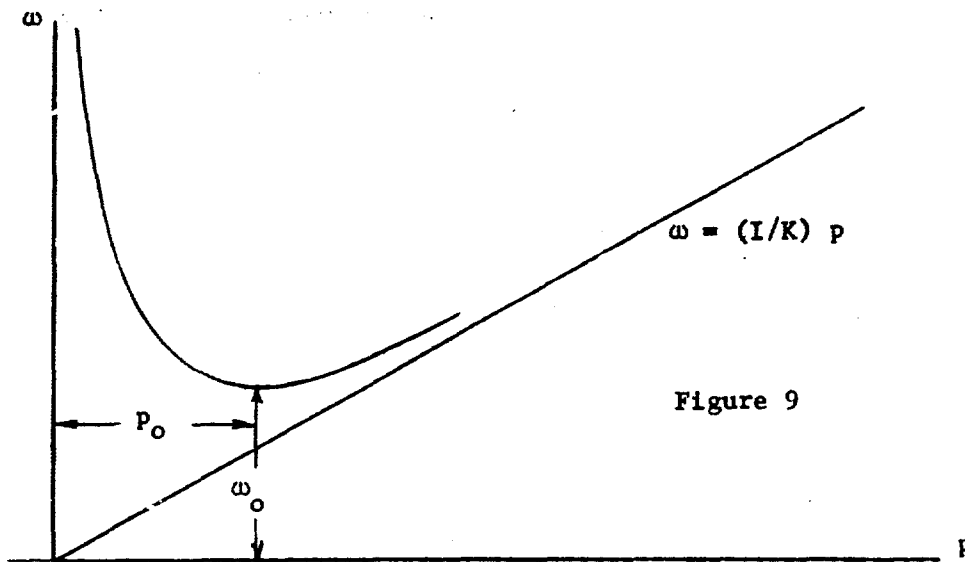
It will be noted that for

$$\omega > \omega_0 \quad (38)$$

that are two real roots of (36) corresponding to two precessional frequencies and stable solutions of (34). On the other hand, for

$$\omega < \omega_0, \quad (39)$$

there exist no real roots of (36); hence, the roots of (36) are complex and the motion is unstable.



From the above it is evident that gyroscopic effects may stabilize an otherwise unstable position of equilibrium.

An illustration of the above occurs for a top spinning on a horizontal table. Here the negative  $C$  arises from the gravity force. For a high enough speed, the top "goes to sleep", with its center of mass above its point of support. As the rate of spin decreases as a result of friction, the position becomes unstable, the top "wobbles", and eventually falls down.

STABILITY OF A ROTATIONALLY ASYMMETRIC ROTOR MOUNTED ON A SHAFT WITH TWO DIFFERENT STIFFNESSES

Let  $I, J, K$  be the three principal moments of inertia of the rotor through its c.m.,  $K$  being the moment of inertia about the rotation axis. We consider a system  $S$  of  $(x, y, z)$  - axes, with unit vectors  $\underline{i}, \underline{j}, \underline{k}$ , rotating relative to fixed space about the  $z$ -axis, with the constant rotor velocity  $\omega$ . Suppose that in the undeflected position the directions  $\underline{i}, \underline{j}$  correspond to the principal axes of inertia of the rotor, as well as to the principal axes of bending stiffness of the shaft crosssection, that is, to the largest and smallest moments of inertia of that section about axes through its center.

Suppose that relative to axes of the rotating system  $S$ , Eqs. 7(1) - 7(4) hold for the deflections of the shaft, but different sets of stiffness constants:

$$F_x = A_1 x + B_1 \mu, \quad T_y = B_1 x + C_1 \mu, \quad (1)$$

$$F_y = A_2 y - B_2 \lambda, \quad T_x = -B_2 y + C_2 \lambda. \quad (2)$$

Here  $x, y; \lambda, \mu$  represent the (small) linear and angular deflections of the rotor shaft at the rotor c.m.,  $z = z_0$ , relative to the rotating axes of  $S$ , while  $F_x, F_y; T_x, T_y$  represent the forces parallel to these axes and the moments about these axes acting on the shaft at the c.m.

Since the system is rotating about the  $z$ -axis, it follows that

$$\frac{d\underline{i}}{dt} = \omega \underline{j}, \quad \frac{d\underline{j}}{dt} = -\omega \underline{i}. \quad (3)$$

To avoid ambiguity, we suppose that the angular displacements  $\lambda, \mu$  are carried out as follows. First the rotation  $\lambda$  is carried out about the vector  $\underline{i}$ ; this rotates the vectors  $\underline{i}, \underline{j}, \underline{k}$  into  $\underline{i}_1, \underline{j}_1, \underline{k}_1$ , given by

$$\underline{i}_1 = \underline{i}, \quad \underline{j}_1 = \underline{j} \cos \lambda + \underline{k} \sin \lambda, \quad \underline{k}_1 = -\underline{j} \sin \lambda + \underline{k} \cos \lambda. \quad (4)$$

Next carry out the rotation  $\mu$  about the vector  $\underline{j}_1$ , leading to

$$\underline{i}_2 = \underline{i}_1 \cos \mu - \underline{k}_1 \sin \mu, \quad \underline{j}_2 = \underline{j}_1, \quad \underline{k}_2 = \underline{i}_1 \sin \mu + \underline{k}_1 \cos \mu. \quad (5)$$

Eqs. (4), (5) apply at any time  $t$ , with the values of  $\lambda, \mu$  corresponding to that time instant. The vectors  $\underline{i}_2, \underline{j}_2, \underline{k}_2$  correspond to the principal directions of the rotor at the time  $t$ , in its deflected (and rotating) position; they are fixed in the rotor  $R$ .

From Eqs. (3), (4), (5) follows that the net rotation vector of  $R$ , relative to fixed space, is given by

$$\underline{\omega} = \omega \underline{k} + \underline{i}_1 \dot{\lambda} + \underline{j}_2 \dot{\mu}. \quad (6)$$

For small  $\lambda, \mu$  Eqs. (4), (5) reduce to

$$\tilde{i}_1 = \tilde{i}, \tilde{j}_1 = \tilde{j} + k\lambda, \tilde{k}_1 = -\tilde{j}\lambda + \tilde{k}, \quad (7)$$

$$\tilde{i}_2 = \tilde{i}_1 - \tilde{k}_1\lambda, \tilde{j}_2 = \tilde{j}_1, \tilde{k}_2 = \tilde{i}_1\mu + \tilde{k}_1, \quad (8)$$

provided powers of  $\lambda, \mu$  higher than the first, be neglected; hence

$$\tilde{i}_2 = \tilde{i} - \mu\tilde{k}, \tilde{j}_2 = \tilde{j} + \lambda\tilde{k}, \tilde{k}_2 = \mu\tilde{i} - \lambda\tilde{j} + \tilde{k}. \quad (9)$$

The inverse of Eqs. (4), (5) take on the same form, but with  $\sin\lambda, \sin\mu$  (or  $\lambda, \mu$ ) replaced by their negatives. Eq. (6) yields, upon retention of terms linear, in  $\lambda, \mu$  only,

$$\begin{aligned} \omega &= \omega(\tilde{k}_2 - \mu\tilde{i}_2 + \lambda\tilde{j}_2) + \dot{\tilde{i}}_2\lambda + \dot{\tilde{j}}_2\mu \\ &= \tilde{i}_2(\lambda - \omega\mu) + \tilde{j}_2(\dot{\mu} + \omega\lambda) + \omega\tilde{k}_2. \end{aligned} \quad (10)$$

The cross product of  $\omega$  by  $\tilde{i}_2, \tilde{j}_2, \tilde{k}_2$  leads to their time rates:

$$\begin{aligned} \frac{d\tilde{i}_2}{dt} &= \omega\tilde{j}_2 - (\omega\lambda + \dot{\mu})\tilde{k}_2, \quad \frac{d\tilde{j}_2}{dt} = -\omega\tilde{i}_2 + (\lambda - \omega\mu)\tilde{k}_2, \quad \frac{d\tilde{k}_2}{dt} = (\dot{\mu} + \lambda\omega)\tilde{i}_2 \\ &\quad - (\lambda - \mu\omega)\tilde{j}_2. \end{aligned} \quad (11)$$

Alternatively equations (11) can be obtained by differentiating (9) and utilizing (3). From Eq. (10) follows for the m.m. of R

$$\tilde{M} = I\tilde{i}_2(\dot{\lambda} - \omega\mu) + J\tilde{j}_2(\dot{\mu} + \omega\lambda) + K\omega\tilde{k}_2, \quad (12)$$

and substituting from (9)

$$\tilde{M} = [I\dot{\lambda} + (K - I)\mu\omega]\tilde{i} + [J\dot{\mu} + (J - K)\omega\lambda]\tilde{j} + K\omega\tilde{k}. \quad (13)$$

Differentiating and utilizing (3), there results

$$\begin{aligned} \dot{\tilde{M}} &= \left[ I\ddot{\lambda} + (K - I - J)\dot{\mu}\omega + (K - J)\omega^2\lambda \right]\tilde{i} \\ &\quad + \left[ J\ddot{\mu} - (K - I - J)\dot{\lambda}\omega + (K - I)\omega^2\mu \right]\tilde{j}. \end{aligned} \quad (14)$$

From Eqs. (1), to (14) and 6(6) the following dynamical equations of rotor motion are obtained:

$$\begin{aligned}
m(\ddot{x} - 2\omega\dot{y} - \omega^2 x) + A_1 x + B_1 \mu &= 0, \\
m(\ddot{y} + 2\omega\dot{x} - \omega^2 y) + A_2 y - B_2 \lambda &= 0, \\
I\ddot{\lambda} + (K - I - J)\dot{\mu}\omega + (K - J)\omega^2 \lambda - B_2 y + C_2 \lambda &= 0, \\
J\ddot{\mu} - (K - I - J)\dot{\lambda}\omega + (K - I)\omega^2 \mu + B_1 x + C_1 \mu &= 0.
\end{aligned} \tag{15}$$

Assuming a solution of Eqs. (15) of the form 7(14) one is led to the determinantal equation

$$\Delta(v) = \begin{vmatrix} \mu & y & \lambda \\ m(v^2 - \omega^2) + A_1 & B_1 & -2m\omega v & 0 \\ B_1 & (Jv^2 + (K - I)\omega^2 + C_1) & 0 & (J + I - K)\omega v \\ 2m\omega v & 0 & (m(v^2 - \omega^2) + A_2) & -B_2 \\ 0 & (K - I - J)\omega v & -B_2 & (Iv^2 + (K - J)\omega^2 + C_2) \end{vmatrix} = 0 \tag{16}$$

Expanding as in 7(16) shows that  $\Delta(v)$  is even in  $v$  as well as in  $\omega$ . Thus only if all four roots  $v^2$  are negative, will the solution of (15) be stable.

As a special case, consider

$$\omega = 0, \tag{17}$$

that is, the case of a non-rotating rotor. The  $(x, y)$ -axes are now fixed. Eqs. (15) separate into one pair of  $(x, \mu)$  equations for the (static) vibrations of the rotor in the  $(x, z)$ -plane, and a pair of  $(y, \lambda)$  equations for similar vibrations in the  $(y, z)$ -plane. The resulting determinantal Eq. (16) factors into two second order determinants, agreeing with Eq. 7(15) for  $\omega = 0$ . The four roots of (16), provided the  $z$ -axis is a position of stable equilibrium, are negative and distinct. By continuity follows that Eq. (16) is stable for a proper range of small speeds:

$$\omega < \omega'. \tag{18}$$

As a more significant check, consider the special case

$$I = J, A_1 = A_2 = A, B_1 = B_2 = B, C_1 = C_2 = 0. \tag{19}$$

of an axially symmetric rotor and shaft -- the case considered in Sec. 7. By proper manipulation of Eqs. (15), and by introducing the variables

$$Z = x + i y, M = \mu - i \lambda, \tag{20}$$

one obtains the fourth order system

$$m (\ddot{Z} + 2i\omega \dot{Z} - \omega^2 Z) + A Z + B M = 0, \quad (21)$$

$$B Z + C M + I \ddot{M} - (K - 2I) i \omega \dot{M} + (K - I) \omega^2 M = 0.$$

Assuming time to enter as the factor

$$e^{i q t} \quad (22)$$

one is led to the equation

$$\begin{vmatrix} A - m(q + \omega)^2 & B \\ B & C + K(\omega^2 + \omega q) - I(q + \omega)^2 \end{vmatrix} = 0 \quad (23)$$

It will be noted that this agrees with 7(29) provided that one replaces

$$\omega \text{ by } \omega, p \text{ by } q + \omega, \quad (24)$$

and notes that factoring out  $e^{i \omega t}$  from  $e^{i p t}$  in 7(31):

$$e^{i p t} = e^{i \omega t} e^{i q t} \quad (25)$$

corresponds to passage from fixed to rotating axes.

The plot of  $q$  vs  $\omega$  can be obtained from Fig. 7 by means of the transformation

$$\omega = \omega, q = p - \omega, \quad (26)$$

that is by moving the points of the curve of Fig. 7 at any constant height  $\omega$  to the left a constant distance equal to  $\omega$ . Again one is led to four values of  $q$  for each  $\omega$ .

The transformation (26) at first sight appears to be at odds with the conclusion that  $\Delta(v)$  in Eq. (16) is even in  $v$ . However, it will be recalled that the fourth order systems (21), 7(26) hold only for the variable  $Z, M$ . The conjugate systems, for  $\bar{Z}, \bar{M}$ , lead not to (26), but to

$$\omega = \omega, q = p + \omega, \quad (27)$$

and this corresponds to shifting the reflection of the curve of Fig. 7 in the horizontal axis, a distance  $\omega$  to the right. Geometrically, different portions of the curve  $\omega = \pm f(p)$ ,  $\omega > 0$ , are skewed;  $\omega = f(p)$  to the left,  $\omega = -f(p)$  to the right, so that equal and opposite  $v$ -roots, still go into equal and opposite roots.

Since the special case (19) is thus always stable, it follows, by continuity, that the system (15) is stable for sufficiently slight rotor and shaft asymmetries.

We turn next to another special case, namely when all the elastic constants vanish:

$$A_1 = A_2 = 0, B_1 = B_2 = 0, C_1 = C_2 = 0. \quad (28)$$

This leads to an unrestrained or free rotor.

Eqs. (15) now separate into  $(x, y)$ -equations which lead to

$$(v^2 - \omega^2)^2 + 4 \omega^2 v^2 = (v^2 + \omega^2)^2 = 0 \quad (29)$$

and to  $(\lambda, \mu)$ -equations from which follows

$$I J v^4 + [I J + (K - I)(K - J)] \omega^2 v^2 + (K - I)(K - J) \omega^4 = 0 \quad (30)$$

It is of interest to point out that Eqs. (29), (30) also follow from (16) when the elastic constants do not vanish, if  $\omega$  is allowed to get large, provided it is assumed that the roots  $v^2$  of  $\Delta(v)$  also get large, so that  $v^2/\omega^2$  remain finite. This assumption is, indeed, valid for (29), (30). Hence the roots of (16) behave like the roots of (29), (30) for large  $\omega^2$ . In the  $(\omega^2, v^2)$ -plane the ratios  $v^2/\omega^2$  correspond to slopes of asymptotes of the curve  $\Delta(v^2, \omega^2) = 0$ .

The repeated roots

$$v^2 = -\omega^2, -\omega^2 \quad (31)$$

lead to solutions  $e^{\pm i \omega t}$ ,  $t e^{\pm i \omega t}$ , which represent uniform rectilinear motion of the c.m., when described in terms of  $x, y$  of the rotating system  $S$ . Eq. (30) can be put in the form

$$x^2 + (1 + a)x + a = 0, \quad x = \frac{v^2}{\omega^2}, \quad a = (K - I)(K - J)/IJ, \quad (32)$$

and factors into

$$(x + 1)(x + a) = 0 \quad (33)$$

thus yielding

$$v^2/a^2 = -1, \quad v^2/\omega^2 = -(K - I)(K - J)/IJ = -a. \quad (34)$$

It will be noted that if  $K$  is the largest or the smallest moment of inertia, the roots (34) lead to stable solutions; on the other hand if

$$I < K < J \quad \text{or} \quad J < K < I \quad (35)$$

then one root of (30) leads to instability.

The above results for (34) agree with the classical Poincaré motion, where the hodograph paths near the principal axes with largest and smallest moments of inertia, are closed curves (on the ellipsoid of inertia), but form two self-intersecting curves through the intermediate axis of inertia. (The root  $v^2 = -\omega^2$

corresponds to a rotation about a principal axis held fixed in space.)

Returning to general values of the elastic constants, the instability in case (35) implies that the solution of (15) is unstable for large enough  $\omega$ . When  $K$  is maximum or minimum, no conclusion can be drawn regarding the stability of (15), except that for large enough  $\omega^2$  the roots are all stable.

It is of interest to find the values of  $\omega^2$  for which  $v^2 = 0$  is a root of (16), since passage from stability to instability may occur by  $v^2$  passing from a negative to a positive value, hence through  $v = 0$ .

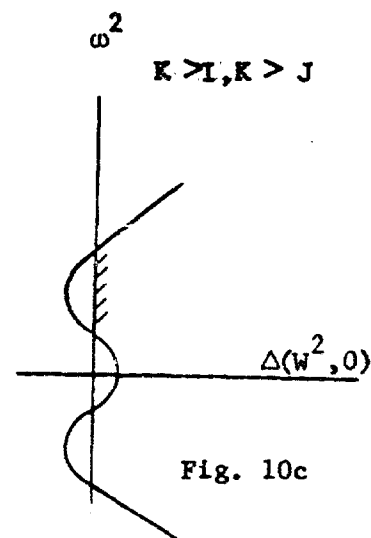
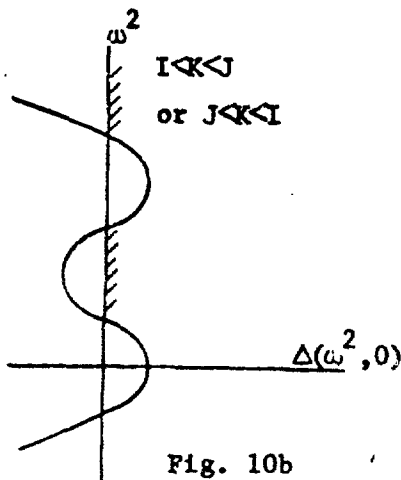
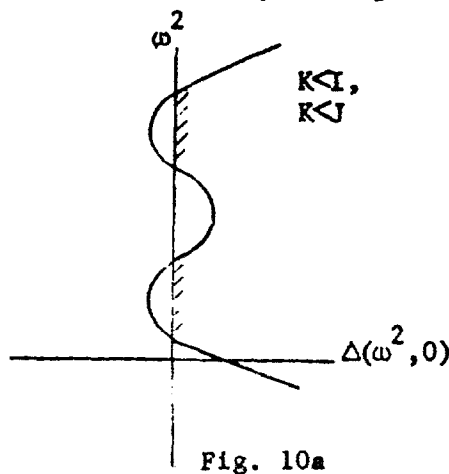
If  $v = 0$ , then  $x, \mu, \lambda,$  become constants, and eqs. (15) separate into an  $(x, \mu)$  - pair leading to

$$\left[ -m\omega^2 + A_1 \right] \left[ (K - I)\omega^2 + C_1 \right] - B_1^2 = 0, \quad (36)$$

and a  $(y, \lambda)$  - pair, resulting in

$$\left[ -m\omega^2 + A_2 \right] \left[ (K - J)\omega^2 + C_2 \right] - B_2^2 = 0. \quad (37)$$

A study of the roots of (36), (37), under the assumption that the  $z$ -axis is a stable position of (static) equilibrium, leads to the conclusion that, when (33) holds, three roots  $\omega^2$  of (16) for  $v = 0$  are positive, one negative. When (35) does not hold, then all the four roots of (16) are positive for  $K > I, K > J$ , or two are positive and two are negative. Plots of  $\Delta(\omega, v)$  for  $v = 0$  are indicated schematically in Fig. 10.



Eq. (16), when expanded in powers of  $v^2$ , yields

$$\Delta(\omega, v) = m^2 I J v^8 + ( \quad ) v^6 + \dots + \Delta(\omega, 0) = 0. \quad (38)$$

The product of the roots of (16) is thus equal to

$$\frac{v_1^2}{1} \frac{v_2^2}{2} \frac{v_3^2}{3} \frac{v_4^2}{4} = \frac{\Delta(\omega^2, 0)}{2IJ} \quad (39)$$

Hence follows that for  $\omega^2$  corresponding to negative  $\Delta(\omega^2, 0)$  in Fig. 10, not all four roots of (15) can be negative. These rotor speed intervals thus correspond to unstable solutions.

Passage from stability to instability may occur not only through  $v^2$  passing from negative to positive values—these occur at the  $\omega^2$ -roots of (36), (37)—but also through two negative roots  $v_1^2, v_2^2$  meeting and receding at right angles in the complex plane. This happens at simultaneous roots of the two equations

$$\Delta(\omega^2, v^2) = 0, \quad \partial \Delta(\omega^2, v^2) / \partial v^2 = 0 \quad (40)$$

In conclusion, for an unsymmetric rotor and shaft, there always exist unstable rotor speed ranges.

As an example, we return to the rotor of Section 6 with the two shaft stiffnesses  $k_1, k_2$  (corresponding to  $A_1, A_2$ ) and the centrally supported rotor, but assume that the latter has two different transverse moments of inertia  $I, J, I \neq J$ . In terms of the constants in Eqs. (1), (2) we have

$$B_1 = B_2 = 0 \quad (41)$$

Eqs. (15) now separate into  $x, y$  equations whose solution has been found in Section 6 and  $\lambda, \mu$  equations leading to the determinant equation

$$f(v^2, \omega^2) = [Jv^2 + (K - I)\omega^2 + C_1] [Iv^2 + (K - J)\omega^2 + C_2] + (K - I - J)^2 v^2 \omega^2 = 0 \quad (42)$$

Since  $f$  is quadratic in  $v^2$  and  $\omega^2$ , Eq. (42), when plotted in the  $(\omega^2, v^2)$ -plane, yields a conic section  $C$ . For large  $\omega^2, v^2$ , the roots of Eq. (42) approach those of (30); thus  $C$  has real asymptotes whose slopes are equal to the right-hand numbers of (34).

The center of C is found by putting

$$\frac{\partial f}{\partial v^2} = 0, \quad \frac{\partial f}{\partial \omega^2} = 0. \quad (43)$$

Solution of the resulting linear equations yields

$$v^2 = v_0^2 = \frac{2a}{(1-a)^2} \left( \frac{C_1}{J} + \frac{C_2}{I} \right) - \frac{(1+a)}{(1-a)^2} \frac{(K-I)C_2 + (K-J)C_1}{IJ},$$

$$\omega^2 = \omega_0^2 = \frac{2}{(1-a)^2} \frac{C_2(K+I) + C_1(K-J)}{IJ} - \frac{1+a}{(1-a)^2} \left( \frac{C_2}{I} + \frac{C_1}{J} \right). \quad (44)$$

Expanding  $f$  in a Taylor's series about the center and factoring as in (32), (33), there results for C

$$\left[ (v^2 - v_0^2) + (\omega^2 - \omega_0^2) \right] \left[ (v^2 - v_0^2) + a(\omega^2 - \omega_0^2) \right] + f(v_0^2, \omega_0^2)/IJ(KI)(K-J) = 0 \quad (45)$$

Schematic plots for C are shown in Figure 11.

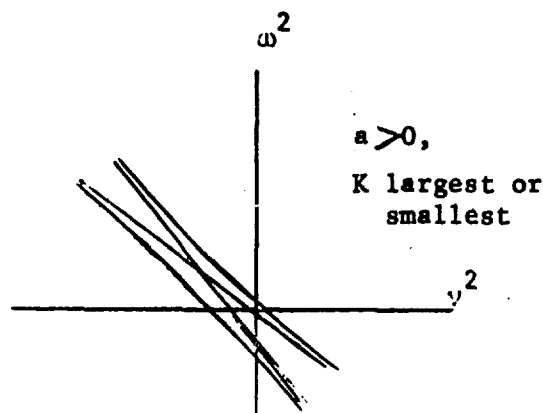


Fig. 11a

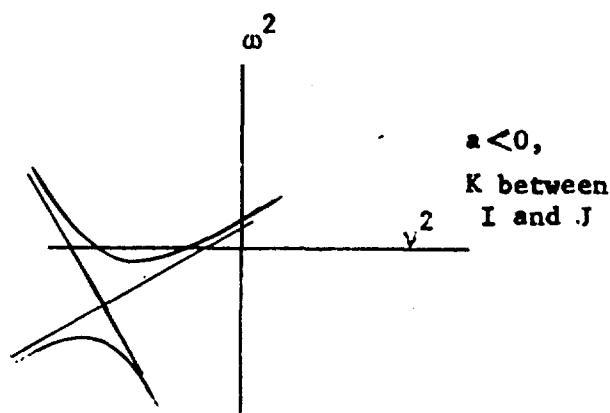


Fig. 11b

Thus far it has been assumed that the  $I, J$  axes of inertia of the rotor are parallel to the principal directions of bending of the shaft. Suppose now that this is no longer the case, and that while Eqs. (3) - (14) apply, the principal directions of bending of the shaft make an angle  $\gamma$  with the principal directions  $\tilde{i}, \tilde{j}$  of rotor moments of inertia. We shall indicate the modifications in the dynamical equations (15) arising from this misalignment of principal axes.

Equations (1), (2) imply an elastic potential energy  $Q$  of the shaft given by (for  $\gamma = 0$ )

$$2Q = (A_1 x^2 + 2B_1 x\mu + C_1 \mu^2) + (A_2 y^2 - 2B_2 y\lambda + C_2 \lambda^2) \quad (46)$$

With  $0 < \gamma < \pi/2$ , the energy  $Q$  is obtained by replacing  $(x, y; \lambda, \mu)$  in (46) by  $(x', y'; \lambda', \mu')$  where  $x', \mu'$  are the linear and angular displacements along (and about) the principal shaft stiffness axes  $x', y'$ , where

$$\begin{aligned} x' &= x \cos \gamma + y \sin \gamma, & y' &= -x \sin \gamma + y \cos \gamma, \\ \lambda' &= \lambda \cos \gamma + \mu \sin \gamma, & \mu' &= -\lambda \sin \gamma + \mu \cos \gamma. \end{aligned} \quad (47)$$

Replacing  $x, \dots, \mu$  in (46) by  $x', \dots, \mu'$ , then expressing the latter in terms of  $x, \dots, \mu$  by means of (47), there results for  $Q$  the quadratic form

$$2Q = \sum_{i,j=1}^4 A_{ij} u_i u_j, \quad A_{ij} = A_{ji}$$

where, to make the summational notation available,  $u_i$ ;  $i = 1, 2, 3, 4$  have been introduced for  $x, y; \lambda, \mu$ , and  $A_{ij}$  are shown in Table I and where the further notational abbreviations

$$c = \cos \gamma, \quad s = \sin \gamma \quad (49)$$

have been made.

TABLE I

$$A_{11} = A_1 c^2 + A_2 s^2, \quad A_{12} = B_1 c^2 + B_2 s^2, \quad A_{13} = (A_1 - A_2) sc, \quad A_{14} = (B_2 - B_1) sc$$

$$A_{22} = c_1 c^2 + c_2 s^2, \quad A_{23} = (B_1 - B_2) sc, \quad A_{24} = (c_2 - c_1) sc$$

$$A_{33} = (A_1 s^2 + A_2 c^2), \quad A_{34} = -(B_1 s^2 + B_2 c^2)$$

$$A_{44} = (c_1 s^2 + c_2 c^2)$$

The dynamical equations replacing (15) are now obtained by replacing the force and torque terms in (15) (i.e. the terms involving  $A_1 \dots e_1$ ) respectively by

$$\frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial u_1} = A_{11}x + A_{12}y + A_{13}\lambda + A_{14}\mu$$

$$\frac{\partial Q}{\partial y} = \frac{\partial Q}{\partial u_2} = A_{21}x + A_{22}y + A_{23}\lambda + A_{24}\mu$$

(50)

$$\frac{\partial Q}{\partial \lambda} = \frac{\partial Q}{\partial u_3} = A_{31}x + A_{32}y + A_{33}\lambda + A_{34}\mu$$

$$\frac{\partial Q}{\partial \mu} = \frac{\partial Q}{\partial u_4} = A_{41}x + A_{42}y + A_{43}\lambda + A_{44}\mu .$$

The resulting equations lead to a corresponding determinantal equation replacing (16), but now with no vanishing elements.

EFFECT OF JOURNAL BEARINGS ON ROTOR STABILITY

Thus far each bearing has been assumed to act as a "fixed" support for the rotor shaft or journal, so that while the rotor shaft rotates freely inside the bearing, the point on the axis of the shaft (or journal) at the bearing center was assumed to undergo no transverse displacements. Even for pre-compressed ball or roller bearings, this assumption is satisfied only approximately, due to the compliance of the bearings. For cylindrical journals, however, where there is definite radial clearance between the journal and bearing surfaces, this assumption will never do, in view of the added freedom of motion of the journal center in its "circle of clearance".

The position, in the middle plane of the bearing, of the journal center  $O_1$  relative to the bearing center  $O$ , for a given load on the rotor (such as the rotor weight), automatically adjusts itself so that both bearings "carry" this load; that is  $O_1$  moves to such a position that the pressure developed by the lubricant in the bearing has a net resultant acting on the journal which is equal and opposite to the applied load. Basically this pressure distribution is due to the shaft "dragging in" the lubricant into the convergent portion of the film.

It has been known for many years that, at times, the rotor vibrations apparently may be induced by the lubricant in the bearings. This can be demonstrated in a laboratory model: by temporarily reducing the lubricant supplied to the bearing, the vibrations diminish. For a horizontal shaft and reasonably high loads these instabilities occur at speeds considerably higher than the critical speed; for lightly loaded bearings or for a vertical shaft they may occur even at low speeds.

To include the effects of journal bearings on rotor stability, it is evident that not only is knowledge needed of the steady state bearing forces corresponding to a fixed position  $O_1$  of the journal center, but also of the transient forces arising from the motion of  $O_1$  near its equilibrium position.

For simplicity, we consider only complete,  $360^\circ$ , self-acting (that is, non-pressurized) journal bearings. We suppose that, as in Fig. 1, the rotor is mounted at the center of the shaft, and that the shaft is supported at its ends by two, exactly similar, perfectly aligned bearings. We assume, furthermore, that the displacement  $OO_1$  of the journal center from the bearing center is the same at each bearing, at all times. Then the same force

$$(F_r, F_t)$$

(1)

will be exerted on the rotor journal by the lubricant in each bearing; here, as shown in Fig. 12,  $F_r$  denotes the radial component of force, measured positive when directed toward the bearing center  $O$ , while  $F_t$  is the force component at right angles to  $OO_1$ .

A section of the bearing (with an exaggerated radial clearance  $C$ ) is shown in Fig. 12. The journal center  $O_1$  can evidently be located only within the "clearance circle" with center at the bearing center  $O$ , and of radius  $C$ . Hence

$$OO_1 = \epsilon C \quad 0 < \epsilon < 1, \quad (2)$$

where  $\epsilon$  is known as the "eccentricity";  $\epsilon = 0$  corresponds to coincident  $O, O_1$ ;  $\epsilon = 1$  to actual contact of journal and bearing surfaces.

Set up axes as in Fig. 12, with the  $y$ -axis in the direction  $OO_1$  of the journal center, steady-state displacement, and let  $\theta$  be measured from the  $y$ -axis. The gap is given by

$$h = C (1 - \epsilon \cos \theta); \quad (3)$$

it is evidently even in  $\theta$ .

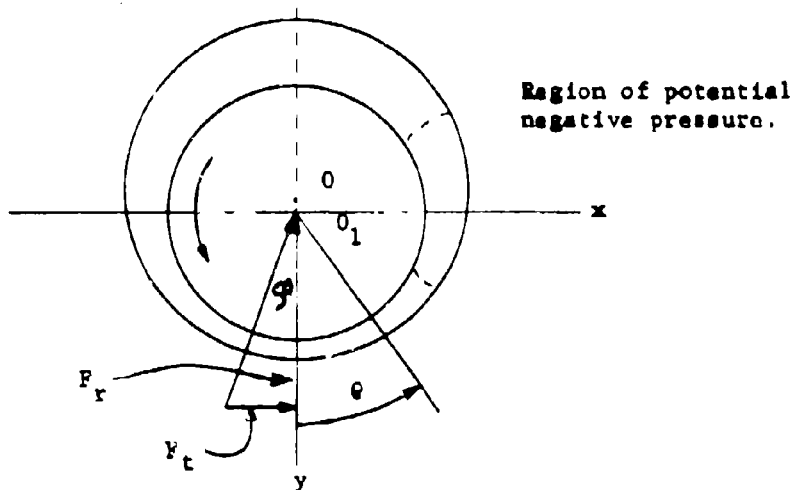


Figure 12

Also shown on Fig. 12 are the force components  $F_r, F_t$  acting on the journal,

as well as the "attitude angle"

$$\varphi = \tan^{-1} F_t / F_r, \quad (4)$$

that is the angle between the applied load, and the displacement vector,  $\overline{OO_1}$ .

The forces  $F_r$ ,  $F_t$  are given by

$$F_r = a \int_{-L/2}^{L/2} ds \int_0^{2\pi} p \cos \theta \, d\theta, \quad (5)$$

$$F_t = -a \int_{-L/2}^{L/2} ds \int_0^{2\pi} p \sin \theta \, d\theta,$$

where  $p$  is the pressure,  $a$  the radius of the journal,  $L$  the axial bearing length,  $z$  the axial distance along the bearing surface from its center line, and  $F_r$  is positive when directed radially inward. The pressure may be found by integrating the Reynolds equation, which for an incompressible lubricant is

$$\frac{1}{\mu} \left[ \frac{\partial}{\partial \theta} \left( \frac{h^3}{a} \frac{\partial p}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( \frac{h^3}{a} \frac{\partial p}{\partial z} \right) \right] = -U \frac{\partial h}{\partial \theta} + 2W, \quad (6)$$

where  $\mu$  = viscosity of lubricant,

$U = a\omega$  = linear velocity of journal due to its rotation,

$W = \dot{h}$  = normal, (i.e. radial) velocity of journal surface, positive when directed toward  $O_1$ .

$W$  vanishes if  $O_1$  is stationary; otherwise it is given by

$$W = -(\dot{u}_r \cos \theta + \dot{u}_t \sin \theta), \quad \dot{u}_r = C\dot{\alpha}, \quad \dot{u}_t = C\dot{\alpha} \quad (7)$$

where  $u_r$  is the radial,  $u_t$  the tangential components of velocity of  $O_1$ , and  $\alpha$  is the angle formed by  $OO_1$  with a fixed direction (on Fig. 12  $\alpha = 0$ ). Hence

$$W = -C\dot{\alpha} \cos \theta - C\dot{\alpha} \sin \theta. \quad (8)$$

Substitution from (3), (8) into (6) yields

$$\frac{1}{\mu} \left\{ \frac{1}{a^2} \frac{\partial}{\partial \theta} \left[ (1 - \epsilon \cos \theta)^3 \frac{\partial p}{\partial \theta} \right] + \frac{\partial}{\partial z} \left[ (1 - \epsilon \cos \theta)^3 \frac{\partial p}{\partial z} \right] \right\} \quad (9)$$

$$= -C\epsilon (\omega - 2\dot{\alpha}) \sin \theta - 2 C\dot{\epsilon} \cos \theta.$$

This equation has to be solved so that  $p$  is periodic in  $\theta$  and satisfies the boundary condition

$$p = p_a \quad \text{at } z = \pm L/2 \quad (10)$$

at the bearing ends, where  $p_a$  is ambient pressure.

Solutions of (9) have been obtained under the assumption that  $p$  is independent of  $z$ :

$$p = p(\theta) \quad (11)$$

an assumption that is valid for a long bearing except near the ends, see references (4), (5). Substituted in (5) they lead to

$$F_r = -K\dot{\epsilon} f_r, \quad F_t = K(\omega - 2\dot{\alpha}) f_t, \quad (12)$$

provided  $F_r$  is measured positively outward, where

$$K = 12\pi \mu a^3 L/C^2, \quad (13)$$

$$f_r = \frac{1}{(1-\epsilon^2)^{3/2}}, \quad f_t = \frac{\epsilon}{(2+\epsilon^2)(1-\epsilon^2)^{1/2}} \quad (14)$$

Consider first stationary  $O_1$ :

$$\dot{\epsilon} = 0, \quad \dot{\alpha} = 0. \quad (15)$$

Eqs. (12) yield

$$F_r = 0, \quad F_t = K\omega f_t(\epsilon). \quad (16)$$

This implies that

$$\theta = \pi/2 \quad (17)$$

The vanishing of  $F_r$  for a stationary  $O_1$  also follows from the symmetry of the film thickness about  $\theta = 0$  and  $\theta = \pi$ , as a result, the pressure increase over the convergent half of the film is matched by an equal pressure decrease over the divergent film half. In equation form:

$$p(\theta) + p(-\theta) = 2p(0) = 2p(\pi); \quad (18)$$

This states that  $p(\theta) - p(0)$  is odd about  $\theta = 0$ , and  $p(\theta) - p(\pi)$  is odd about  $\theta = \pi$ . (That  $p(0) = p(\pi)$  follows by putting  $\theta = \pi$  in the first eq. (18) since  $p(\theta)$  is periodic of period  $2\pi$ ). Indeed, eq. (18) follows from eq. (9) for  $\dot{e} = 0$ , by noting that the right-hand member of (9) is odd in  $\theta$ .

Likewise, the coefficient of  $\dot{e}$  in Eq. (9) is even in  $\theta$ , and the resulting contribution to  $F_r$ ,  $F_t$  in (5) leads to a vanishing  $F_t$ .

The same considerations of evenness and oddness of  $p$  carry over to solutions of Eq. (9) for bearings of finite axial length  $L$ . Eqs. (12) are valid for them, too, but with a different  $K$ , and with  $f_r$ ,  $f_t$  functions of  $e$  and  $L/2a$ .

Actually, the above prediction (17) does not agree with observed positions of  $O_1$ . For a vertical load of different magnitudes and at different speeds  $\omega$ , the equilibrium positions of  $O_1$  in its circle of clearance, tend to cluster near a locus schematically shown in Figure 13, and approximating a semi-ellipse. At most, (17) holds only for small  $e$ , that is for light loads, or very high speeds.

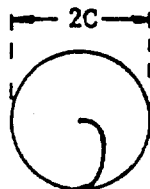


Figure 13

The explanation of the above discrepancy between theory and tests is due to the negative pressures which are allowed in the solution of Eq. (9), but which cannot be realized in a liquid lubricant. If the pressure is decreased in oil, it will tend to cavitate and release dissolved air and oil vapor. Therefore, Eq. (9) must be solved subject to the condition

$$p \geq 0, \quad (19)$$

and putting

$$p = 0 \text{ over } A_c \quad (20)$$

where  $A_c$  denotes the cavitating area. Many shapes of  $A_c$  may be found. A "minimum loss" principle, Cameron and Wood, reference (6), have proposed the condition of vanishing normal pressure gradient at the edge of  $A_c$ :

$$\frac{\partial p}{\partial n} = 0 \text{ at boundary of } A_c. \quad (21)$$

On Figure 12 the cavitating area  $A_c$  is shown schematically between the broken lines on the divergent film half. With (20) holding on  $A_c$ , Eq. (18) fails to apply for  $\dot{\alpha} = 0$ ,  $\dot{e} = 0$ , and a non-zero  $F_r$  results.

For a long bearing, if Eq. (11) is assumed,  $A_c$  is bounded by two lines of constant  $\theta$ , and these are determined by Eqs. (20), (21). For finite  $L$ , the determination of  $A_c$  may require trial and error methods and special programming on high-speed computing machines.

Physically, the boundary  $A_c$  may vary with  $p_a$  in Eq. (10) and with the nature of the liquid lubricant, its vapor pressure, and its dissolved gases.

While (9) remains linear in  $p$  outside  $A_c$ , the introduction of  $A_c$  whose boundary may change with the right-hand number of (9), renders the problem non-linear, and superposition of solutions no longer applies.

It will be noted that while the right-hand member of Eq. (9) involves  $\omega$ ,  $\dot{\alpha}$  and  $\dot{e}$ , the two former occur in the combination  $\omega - 2\dot{\alpha}$ . Therefore, the integration, even with the cavitating conditions, Eqs. (19) - (21), can be confined to

$$\dot{\alpha} = 0, \quad (22)$$

and the results used to obtain  $F_r$ ,  $F_t$  for any  $\dot{\alpha}$ , by replacing  $\omega$  with

$$\omega \rightarrow \omega - 2\dot{\alpha} = \omega g, \quad g = 1 - 2\dot{\alpha}/\omega. \quad (23)$$

Assuming  $\dot{\alpha}$  to vanish, and using dimensional analysis considerations, one obtains, for a particular ratio of  $L/2a$ ,

$$F_r = -l \mu \omega f_r(e, e'), \quad F_t = l \mu \omega f_t(e, e'), \quad (24)$$

where  $f_r$ ,  $f_t$  are dimensionless measures of  $F_r$ ,  $F_t$ ,  $l$  has the dimensions of a length so that  $l \mu \omega$  has the units of a force, and  $e$  has been rendered dimensionless

by introducing

$$\tau = \omega t, \quad \epsilon' = \dot{\epsilon}/\omega = d\epsilon/d\tau. \quad (25)$$

It may be shown that the length  $l$  can be put in the form

$$l = a^3 L/C^2, \quad (26)$$

as in Eqs. (12), (13). It follows now that for  $\dot{\alpha} \neq 0$ ,

$$\begin{aligned} F_r &= -l\mu (\omega - 2\dot{\alpha}) f_r (\epsilon, \epsilon'/g), \\ F_t &= l\mu (\omega - 2\dot{\alpha}) f_t (\epsilon, \epsilon'/g). \end{aligned} \quad g = 1 - 2\alpha' \quad (27)$$

For small  $\epsilon$  cavitation is not likely to occur. If  $L/a$  is large, Eqs. (12) - (14) may form a fair approximation. Neglecting powers of  $\epsilon$ ,  $\dot{\epsilon}$  higher than the first, they yield,

$$F_r = -K\dot{\epsilon}, \quad F_t = K(\omega - 2\dot{\alpha}) \epsilon/2 = \frac{K\omega\epsilon}{2} - K\dot{\alpha}\epsilon. \quad (28)$$

Note on Figure 14, that  $C\dot{\epsilon}$ ,  $C\epsilon\dot{\alpha}$  are the radial and tangential velocity components of  $O_1$ . Hence, the force exerted by both bearings on the shaft, in rectangular components, is

$$F_x = -2f\dot{\xi} - f\omega\eta, \quad F_y = -2f\dot{\eta} + f\omega\xi, \quad f = \frac{K}{C} \quad (29)$$

where  $\xi, \eta$  are the components of  $OO_1$  in the x- and y- directions.

Suppose now that the shaft is infinitely rigid. Then the equations of motion are

$$m \ddot{\xi} = F_x = -f\dot{\xi} - f\omega\eta, \quad m \ddot{\eta} = F_y = -2f\dot{\eta} + f\omega\xi \quad (30)$$

Introducing

$$\zeta = \xi + i\eta \quad (31)$$

and assuming time to enter in the form  $e^{\lambda t}$ , one is led to

$$m\lambda^2 + 2f\lambda + i\omega = 0, \quad (32)$$

an equation which by the tests of Section I is always unstable.

Suppose now that a radial bearing force be added to the above so that (29) is changed into

$$F_x = -2f\xi - f\omega\eta - k_b\xi, \quad F_y = -2f\eta + f\omega\xi - k_by \quad (33)$$

where  $k_b$  is the radial bearing "spring stiffness" constant. Equation (32) becomes

$$m\lambda^2 + 2f\lambda + (k_b + i\omega) = 0. \quad (34)$$

This equation agrees with 5(3) provided, in the latter, one puts

$$f_1 = f, \quad k = k_b. \quad (35)$$

Hence, applying 5(6), we find that the rotor is stable for speeds

$$\omega < 2\omega_0, \quad \omega_0 = \sqrt{\frac{k_b}{m}}, \quad (36)$$

that is, for speeds below double the critical, defined as the frequency of vibration of the rotor mass against the bearing (radial) stiffness. For all higher rotor speeds, the rotor motion is unstable.

Suppose next that the shaft is not infinitely rigid but possesses a finite stiffness constant,  $k$ . Then, to the given center deflection  $(\xi, \eta)$  is added a further deflection of the rotor center, given by

$$x = F_x/k, \quad y = F_y/k, \quad (37)$$

resulting in a more complex set of dynamical equations than (30). The exponential solutions  $e^{\lambda t}$  of these equations are determined by the cubic

$$2\lambda^3 + \left(\frac{k_b + k}{k} - i\omega\right)\lambda^2 + 2\lambda\frac{k}{f} + \frac{k}{m}\left[\left(\frac{k_b}{k}\right) - i\omega\frac{k}{m}\left(\frac{k_b}{k} - i\omega\right)\right] \quad (38)$$

It is shown in Reference (7) that this equation is stable for

$$\omega < 2\omega_0 \quad (39)$$

and unstable for higher rotor speeds, where  $\omega_0$  is defined by

$$\omega_0^2 = \frac{1}{\left(\frac{1}{k} + \frac{1}{k_b}\right)m} = \frac{kk_b}{(k + k_b)} \frac{1}{m}. \quad (40)$$

This critical  $\omega_0$  is the vibration frequency of the rotor mass against the joint bearing and shaft flexibilities (in series).

The transition to instability occurs as the rotor speed  $\omega = 2\omega_0$ . As  $\omega$  passes through this value, one of the roots of (38) can be shown to cross over into the half-plane  $\text{Re}(\lambda) > 0$ , the crossing taking place at

$$\lambda = i\omega_0 \quad (41)$$

corresponding to a circular whirl of angular frequency equal to the critical frequency, even though the rotor speed is twice the critical.

For  $\omega > 2\omega_0$  a root of (38) representing an expanding whirl results, with the frequency of whirl increasing, but slowly, with  $\omega$ .

If  $k_b$  is allowed to approach zero, but  $k$  to remain finite, then  $\omega_0$  in Equation (41) approaches zero, and rotor instability at all speeds results. This result agrees with earlier studies of bearing-rotor motion, based on Equations (12, (14), Reference (8)).

In Equations (28)-(41) we have considered motion of  $O_1$  about the bearing center  $O$  for a journal free from external load. Actually,  $k_b$  is likely to vanish for small  $\epsilon$ , judging from Figure 13, and some external load is required to bring about cavitation on the expanding film side and generate a non-zero  $F_r$  and  $k_b$ . With a constant external load, Equation (33) is applicable, with  $\xi, \eta$  representing the displacements of the journal center from the equilibrium position (see Figure 14), and  $F_x, F_y$  the added bearing forces due to this added displacement. At the same time  $\epsilon$  must be sufficiently small to justify the approximations in Equation (28) to Equations (12) and (14).

The above theory historically represented the first successful theoretical treatment of bearing induced whirl (see Reference (7)), and it was obtained at a time when no solutions of Equations (9) and (19) were available. By increasing

$k_b$  with  $\epsilon$ , one obtains a higher  $\omega_0$  in Equation (40), thus explaining the increased stable speed range (39), with higher static loads. Yet a more sound treatment should be based on solution of Equation (9) with cavitation.

Such solutions have been carried out for various  $L/2a$  (see Reference 9), so that the functions  $f_r$ ,  $f_t$  and their derivations at  $\epsilon' = 0$  are available. We outline the small displacement stability calculation, based on them and on Equation (27), for sizeable  $\epsilon$ .

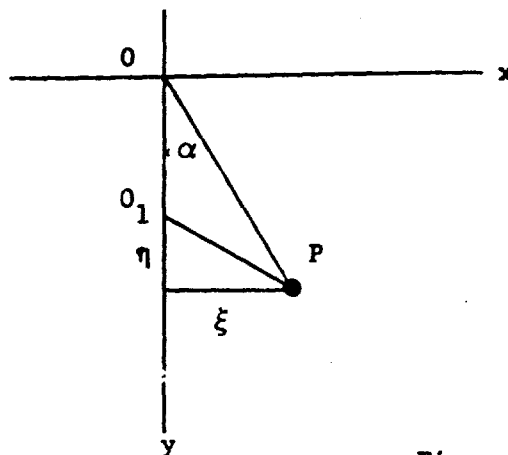


Figure 14

On Figure 14 let  $O_1$  on the  $y$ -axis be the equilibrium position of the journal center for a given static load,  $\xi$ ,  $\eta$  the small displacement  $O_1P$  of the journal center from  $O_1$ , and  $d\epsilon$ ,  $d\alpha$  the corresponding increment in  $\epsilon$ ,  $\alpha$ . Equations (27) for small  $d\epsilon$ ,  $d\alpha$ ,  $d\dot{\epsilon}$ ,  $d\dot{\alpha}$  yield

$$\begin{aligned} dF_r &= l\mu\omega \left( \frac{\partial f_r}{\partial \epsilon} d\epsilon + \frac{\partial f_r}{\partial \epsilon'} d\epsilon' \right) - 2d\dot{\alpha}f_r \\ dF_t &= l\mu\omega \left( \frac{\partial f_t}{\partial \epsilon} d\epsilon + \frac{\partial f_t}{\partial \epsilon'} d\epsilon' \right) - 2d\dot{\alpha}f_t, \end{aligned} \quad (42)$$

where  $f_r$  and  $f_t$  and their derivatives are evaluated at  $\epsilon' = 0$ . In rectangular components, Equation (42) yields for the added forces exerted on both journals.

$$\begin{aligned}
X &= K' \left( -\frac{\omega f_r}{\epsilon} \xi - \frac{2f_t}{\epsilon} \xi + \omega \frac{\partial f}{\partial \epsilon} \eta + \frac{\partial f}{\partial \epsilon'} \eta \right), \\
Y &= K' \left( -\frac{\omega f_t}{\epsilon} \xi + \frac{2f_r}{\epsilon} \xi - \frac{\omega \partial f}{\partial \epsilon} \eta - \frac{\partial f}{\partial \epsilon'} \eta \right),
\end{aligned} \tag{43}$$

$$K' = 2\mu La^3/C^3.$$

Attention must be called to the fact that  $dF_r$ ,  $Y$  have opposite sign conventions, and to the use of the relations,

$$d\alpha = \xi/C\alpha\epsilon, \quad d\dot{\alpha} = \xi/C\alpha\epsilon \tag{44}$$

in obtaining (43). Equations (44) definitely excludes the small  $\epsilon$  case.

We now turn to the solution of the dynamical rotor equations.

$$m\ddot{\xi} = X, \quad m\ddot{\eta} = Y, \tag{45}$$

considering first a rigid rotor shaft. We assume a solution of Equations (43), (45) varying with time as  $e^{\lambda t}$ , and put

$$\lambda = v\omega_0, \quad \omega = s\omega_0 \tag{46}$$

where  $\omega_0$  has the dimensions of  $\omega$ , (1/sec), but as yet undetermined, so that  $v$ ,  $s$  are dimensionless. There results

$$\Delta(v) = \begin{vmatrix} sf_r + 2vf_t + E\epsilon & s\frac{\partial f_r}{\partial \epsilon} + v\frac{\partial f_t}{\partial \epsilon'} \\ -sf_t + 2f_r v & s\frac{\partial f_r}{\partial \epsilon} + v\frac{\partial f_r}{\partial \epsilon'} + E \end{vmatrix} = 0 \tag{47}$$

where

$$E = \frac{m\omega_0^2 v^2}{K'} = \frac{mC^3 \omega_0^2 v^2}{2\mu La^3}. \tag{48}$$

Equation (47) can be shown to be stable for low  $s$  and unstable for high enough  $s$ . The transition to instability occurs by  $v$  crossing the pure imaginary axis, at  $v = \pm iv_0 \neq 0$ . By substituting  $v = iv_0$  in Equation (47) and equating to zero the real and imaginary parts separately, two equations result. The second

equation yields

$$\frac{f_r}{f_t} = f_1(\epsilon) = - \frac{(f_r \frac{\partial f_r}{\partial \epsilon} + f_t \frac{\partial f_t}{\partial \epsilon}) + 2(f_t \frac{\partial f_r}{\partial \epsilon} - f_r \frac{\partial f_t}{\partial \epsilon})}{(2f_t + \epsilon \frac{\partial f_r}{\partial \epsilon})} \quad (49)$$

and applying this to the first one, one obtains

$$\frac{v^2}{2s^2} = f_2(\epsilon) = - \frac{\left[ f_r - \epsilon f_1(\epsilon) \right] \frac{\partial f_t}{\partial \epsilon}}{-f_t \left[ \frac{\partial f_r}{\partial \epsilon} - f_1(\epsilon) \right]} \div \frac{\left[ f_t \frac{\partial f_t}{\partial \epsilon} \right]}{\left[ f_r \frac{\partial f_r}{\partial \epsilon} \right]} \quad (50)$$

Dividing Equation (49) by (50) and recalling (48), there results

$$s = \frac{f_2}{4f_1} = \frac{\mu C^3 \omega_o^2}{\mu L a^3} \quad (51)$$

Now choose  $\omega_o$  so that  $s = 1$ . Then Equations (46) and (51) lead to

$$\omega^2 = \omega_o^2 = \frac{4\mu L a^3}{m C^3} \frac{4f_1(\epsilon)}{f_2(\epsilon)} \quad (52)$$

while Equations (46) and (50) yield

$$\lambda^2 = -2f_2(\epsilon) \omega_o^2 \quad (53)$$

Equation (52) determines the critical speed,  $\omega = \omega_o$ , at which rotor motion becomes unstable, while Equation (53) yields the frequency of the insipient whirl at that speed.

The above theory can be extended to include a finite shaft stiffness coefficient  $k$ , by properly changing the definition of  $E$ . (see Reference 9)

Turning to a brief consideration of compressible lubricants, it is evident that cavitation does not occur since a gas can expand to fill the divergent film half. A  $(p, V)$  curve of the form schematically indicated in Figure 15 results, where  $V$  is the specific volume. It is evident that equal and opposite volume changes  $\pm \Delta V$ , lead to unequal pressure changes  $(\Delta p)^+$ ,  $(\Delta p)^-$ . Thus more pressure

increase is obtained on the convergent film half on Figure 12, than pressure decrease on the divergent film half, and Equation (5) leads to a non-zero  $F_I$ .

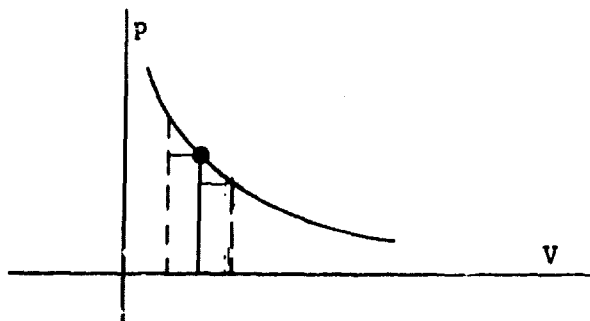


Figure 15

The Reynolds Equation (9) is replaced by

$$\frac{\partial}{\partial z} \left( h^3 \rho \frac{\partial p}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( h^3 \rho \frac{\partial p}{\partial \theta} \right) = 6\mu \left[ -U \frac{\partial(h\rho)}{\partial x} + 2\rho W + 2h \frac{\partial p}{\partial t} \right] \quad (54)$$

Assuming an isothermal  $(p, \rho)$  curve simplifies Equation (54) to

$$\frac{\partial}{\partial z} \left( h^3 p \frac{\partial p}{\partial z} \right) + \frac{\partial}{\partial \theta} \left( h^3 p \frac{\partial p}{\partial \theta} \right) = 6\mu \left[ -a\omega \frac{\partial(h\rho)}{\partial x} + 2pW + 2h \frac{\partial p}{\partial t} \right] \quad (55)$$

This is non-linear in  $p$ . It no longer involves  $\dot{\alpha}$ ,  $\omega$  in the simple form  $\omega - 2\dot{\alpha}$ , as in Equations (9) and (12). For stability calculations near equilibrium positions, solutions varying with time as  $e^{\lambda t}$ , require further solution of Equation (55), due to the term  $\partial p / \partial t$  on the right.

EFFECT OF DIFFERENT BEARING SUPPORT STIFFNESS CONSTANTS

Thus far, the bearing supports have been assumed to be rigid. We now consider the effect of resilience or flexibility of these supports on the stability of the rotors and shafts considered in Sections VI - VIII. The inertia of the supports will be neglected.

The bearing supports generally possess different stiffness constants in two mutually perpendicular, fixed directions, say, the horizontal and vertical directions. Suppose that as in Section VIII, the shaft, too, has different bending stiffness constants in two directions; these rotate with the shaft. The net resilience of the shaft and bearing supports turn out to vary with the rotation angle

$$\tau = \omega t \quad (1)$$

whether it is expressed along fixed or rotating axes. One is thus led to linear differential equations of rotor motion with periodic coefficients. For the rotor of Section VI an 8th order system with periodic coefficient results; for special cases this separates into two 4th order systems, or reduces to a single 4th order system. For the rotor of Section VI, the translational motion leads to the system

$$\begin{aligned} \mu x + (1+\sigma+\rho \cos 2\tau) \frac{d^2 x}{d\tau^2} + (\rho \sin 2\tau) \frac{d^2 y}{d\tau^2} &= 0, \\ \mu y + \rho \sin 2\tau \frac{d^2 x}{d\tau^2} + (1-\sigma-\rho \cos 2\tau) \frac{d^2 y}{d\tau^2} &= 0, \end{aligned} \quad (2)$$

where  $\mu, \rho, \sigma$  are proper constants. The rotational displacements (with unequal  $I, J$ ), leads to another 4th order system. We consider the stability of solutions of (2).

As stated in Section I there exists special solutions of linear differential equations with periodic coefficients such as (2) of the form

$$P(\tau) e^{\lambda \tau} \quad (3)$$

where  $P$  is periodic of period  $\Delta\tau = \pi$  and  $\lambda$  is a constant. Four such solutions exist for the 4th order system (2), each with its own exponential  $e^{\lambda\tau}$ , and general solution is a linear combination of these four. For stability of the motion all four  $\lambda$ 's must be in the half-plane

$$\operatorname{Re}(\lambda) \leq 0 \quad (4)$$

If any one of the  $\lambda$ 's fails to satisfy (4), then the general solution of Eq. (2) will become infinite as  $t$  increases.

The stability conditions (4) are similar to those for linear differential equations with constant coefficients; the difference lies in the great difficulty of obtaining the values of  $\lambda$ .

Since

$$e^{\lambda(\tau+\pi)} = Ce^{\lambda\tau}, \quad C = e^{\lambda\pi}, \quad (5)$$

the solutions (3) have the property of being multiplied by a constant  $C$  when  $\tau$  increases by the period  $\pi$ . The requirement (4) leads to the condition that all four  $C$ 's lie within or on the unit circle

$$|C| \leq 1. \quad (6)$$

The derivation of (2) proceeds as follows. Denote by  $\underline{i}_0, \underline{j}_0$  unit vectors along the (fixed)  $(x, y)$ -axes of Fig. 1, and by  $\underline{i}, \underline{j}$  unit vectors rotating with the shaft, and corresponding to the directions of the principal moments of inertia of its cross section. A force

$$\underline{F} = F_x \underline{i}_0 + F_y \underline{j}_0 = F_x (\underline{i} \cos \tau - \underline{j} \sin \tau) + F_y (\underline{i} \sin \tau + \underline{j} \cos \tau) \quad (7)$$

applied to the rotor, will produce a deflection due to shaft bending

$$f_1 (F_x \cos \tau + F_y \sin \tau) \underline{i} + f_2 (-F_x \sin \tau + F_y \cos \tau) \underline{j} \quad (8)$$

and a deflection

$$f_3 F_x \underline{i}_0 + f_4 F_y \underline{j}_0 \quad (9)$$

due to bearing support deflections, where  $f_1, f_2$  are the flexibilities (i.e. reciprocals of stiffnesses) of the shaft in the  $i, j$  - directions;  $f_3, f_4$  the flexibilities of the bearing supports in the  $\tilde{i}_0, \tilde{j}_0$  directions. We assume that  $f_1 > f_2, f_3 > f_4$ . Adding the displacements (8) (9), expressing  $i, j$  in (7) in terms of  $\tilde{i}_0, \tilde{j}_0$ , and introducing the notation

$$f_m = (f_1 + f_2 + f_3 + f_4)/2, \quad \rho = (f_1 - f_2)/2f_m, \quad \sigma = (f_3 - f_4)/2f_m \quad (10)$$

where  $f_m$  is the mean, joint flexibility of the shaft and bearing supports, there results for the net rotor center deflection  $x, y$

$$x/f_m = (1 + \sigma + \rho \cos 2\tau) F_x + (\rho \sin 2\tau) F_y, \quad (11)$$

$$y/f_m = (\sin 2\tau) F_x + (1 - \sigma - \rho \cos 2\tau) F_y.$$

If the force  $F_x, F_y$  is due to the rotor inertia, then

$$F_x = -m \frac{d^2 x}{dt^2}, \quad F_y = -m \frac{d^2 y}{dt^2}. \quad (12)$$

Then, replacing  $F_x, F_y$  in (11) from (12) introducing  $\tau$  as in (1), and putting

$$\omega_o^2 = \frac{1}{mf_m}, \quad \mu = \frac{\omega_o^2}{2} \quad (13)$$

one arrives at Eqs. (2).

Before considering the solution of (2) proper, we first note the special cases  $\rho = 0$ , and  $\sigma = 0$  of equal shaft or bearing support flexibilities. In the former case

$$\rho = 0, \quad f_1 = f_2 = f_s, \quad (14)$$

the trigonometric terms in (2) disappear, the periodic terms  $P(\tau)$  in (3) reduce to constants, the  $\lambda$ 's turn out to be pure imaginary and correspond to sinusoidal solutions in  $x, y$  of frequencies  $\omega_1, \omega_2$  respectively where

$$\omega_1^2 = \frac{1}{m(f_2 + f_3)}, \quad \omega_2^2 = \frac{1}{m(f_s + f_4)} \quad (15)$$

In the latter case,

$$\sigma = 0, f_3 = f_4 = f_b; \quad (16)$$

by using rotating axes, eqs. (2) may now be reduced to Eqs. 6(7), but with  $k_1, k_2$  replaced by

$$\frac{1}{k_1} = f_1 + f_b, \quad \frac{1}{k_2} = f_2 + f_b. \quad (17)$$

Eqs. 6(13) yields the unstable range.

$$\frac{1}{m(f_b + f_3)} < \omega^2 < \frac{1}{m(f_b + f_4)}; \quad (18)$$

while for  $\omega^2$  outside this range the motion is stable. In terms of  $\sigma, \rho$  Eqs. (15) - (18) become

$$\rho = 0; \quad \mu = (1 - \sigma), \quad \mu = 1 + \sigma, \quad (18)$$

$$\sigma = 0; \quad 1 - \rho < \mu < 1 + \rho. \quad (19)$$

Summarizing, it is seen that for  $\rho = 0$  the constant bearing flexibility may be added to each shaft flexibility.

For general  $\sigma, \rho$  periodic coefficients persist in Eqs. (2) whether fixed or rotating axes are used.

Following Floquet, an algebraic equation for the four characteristic constants  $C$  (see Eqs. (3) - (6)) can be formulated if four (linearly independent) solutions of (2) are available. It is given by the fourth-order algebraic equation

$$|u_{ij}(\pi) - C \delta_{ij}| = 0, \quad (20)$$

where  $||\delta_{ij}||$  is the fourth-order, unit matrix ( $\delta_{ij} = 1$  for  $i = j$ ,  $\delta_{ij} = 0$  for  $i \neq j$ ), and

$$u_{i1} = x_i(\tau), \quad u_{i2} = x_i'(\tau), \quad u_{i3} = y_i(\tau), \quad u_{i4} = y_i'(\tau); \quad i = 1, 2, 3, 4 \quad (21)$$

are four solutions of (2) whose initial values are equal to the  $i$ -th row of the unit matrix  $\delta_{ij}$ . However, since separate integrations of (2) over the interval

$0 < \tau < \pi$  have to be carried out for each  $\sigma$ ,  $\mu$ , procedures based on (20) appear economically prohibitive, even with the aid of high speed computing machines.

Expansion of the solutions of (2) in powers of  $\rho$  is a possible method of integration of Eqs. (2), provided both  $\lambda$  and  $P(\tau)$  in (3) are expanded in powers of  $\rho$ . The convergence for sizable  $\rho$  is very slow, however.

Also slow is the convergence of the "classical" method, due to A. V. Hill, based on the expansion of  $P(\tau)$  in (3) in Fourier series in  $\tau$ . A significant improvement in its convergence was effected by Foote, Poritsky, and Slade, in Ref. (10); we proceed with a qualitative description of this method. (See Ref. 11)

First, we simplify Eqs. (2) by introducing, as in Sections III-V, VII, VIII

$$Z = x + iy, \bar{Z} = x - iy, i = \sqrt{-1}. \quad (22)$$

One obtains from Eqs. (2)

$$\begin{aligned} \ddot{Z} + \mu Z + (\sigma + e^{2i\tau}) \ddot{\bar{Z}} &= 0, \\ \ddot{\bar{Z}} + \mu \bar{Z} + (\sigma + e^{-2i\tau}) \ddot{Z} &= 0. \end{aligned} \quad (23)$$

Setting

$$Z = e^{\lambda\tau} \sum A_n e^{ni\tau}, \quad \bar{Z} = e^{\lambda\tau} \sum B_n e^{ni\tau}; n = \pm 1, \pm 3, \dots \quad (24)$$

substituting (24) into (2) and equating net coefficients of each exponential to zero, one arrives at the equations

$$\begin{aligned} \rho B_{n-2} + \left[ 1 + \frac{\mu}{(\lambda + ni)^2} \right] A_n + \sigma B_n &= 0 \\ \sigma A_n + \left[ 1 + \frac{\mu}{(\lambda + m)^2} \right] B_n + \rho A_{n+2} &= 0 \end{aligned} \quad (25)$$

for the coefficients  $A_n, B_n$ , where  $n = \pm 1, \pm 3, \dots$ . By equating determinants of central blocks to zero, one obtains

$$\Delta_n(\lambda) = 0. \quad (26)$$

The limit of the roots of (26) as  $n$  gets large, yield the values of  $\lambda$ .

First, it will be noted that renaming the (dummy) summation index  $n$  in (24) as  $n+2$  or  $n-2$ , is equivalent to replacing  $\lambda$  by  $\lambda-2i$ ,  $\lambda+2i$  respectively. Hence, to each value of  $\lambda$  in (3), the periodic array  $\lambda+2ni$  also furnishes possible values; they all lead to the same value of  $C$  in (5) since  $e^{2\pi i} = 1$ . Next note that changing  $t$  into  $-t$  and interchanging  $Z$ ,  $\bar{Z}$  leaves (2) invariant. Hence the four  $\lambda$ 's are arranged in two pairs,  $\pm\lambda_1$ ,  $\pm\lambda_2$ . Finally, from (25) it will also be seen that  $\lambda = \pm ni$ ;  $n=1,3,\dots$  are fourth order poles of  $\Delta_n(\lambda)$ . Now a function of  $\lambda$  having all the above properties is given by

$$f(\lambda) = \frac{(\sinh^2 \frac{\lambda\pi}{2} + \sinh^2 \frac{\lambda_1\pi}{2})(\sinh^2 \frac{\lambda\pi}{2} - \sinh^2 \frac{\lambda_2\pi}{2})}{\cosh^4(\pi\lambda/2)} \quad (27)$$

It is therefore to be expected that the ratio of  $\Delta_n$  by a proper function of  $n$  approaches  $f(\lambda)$  as  $n \rightarrow \infty$ .

The function  $off(\lambda)$  can be expressed as a trigonometric function of

$$(a\pi/2), \quad \lambda = ai, \quad (28)$$

and its numerator as a quadratic

$$x^2 - x(K_1 - K_2 + 1) + K_1 = 0, \quad (29)$$

where

$$x = \sinh^2 \frac{\lambda\pi}{2} = (C-1)^2/4C \quad (30)$$

$$K_1 = \sinh^2 \frac{\lambda_1\pi}{2} \sinh^2 \frac{\lambda_2\pi}{2}$$

$$K_2 = \cosh^2 \frac{\lambda_1\pi}{2} \cosh^2 \frac{\lambda_2\pi}{2}.$$

Thus the  $C$ 's and  $\lambda$ 's can be found once  $K_1$ ,  $K_2$  have been obtained.

The convergence of (the roots of)  $\Delta_n(\lambda)$  to (the roots of)  $f(\lambda)$  is very slow. This can be demonstrated from the convergence of the infinite products in

$$\cos \frac{\pi u}{2} = \prod_{k=1,3,\dots}^{\infty} (1 - \frac{u^2}{k^2}), \quad \sin \frac{\pi u}{2} = \frac{\pi x}{2} \prod_{k=2,4,\dots}^{\infty} (1 - \frac{u^2}{k^2}) \quad (31)$$

to their limits. In Reference (10),  $K_1$ ,  $K_2$  are computed as limits of certain sequences of  $K_{1n}$  to  $K_1$ , and  $K_{2n}$  to  $K_2$  is speeded up by means of "convergence factors"

which are basically the ratio of the left-hand members of (31) to their finite product sums. For details the reader is referred to the reference proper.

Figure 16 shows the unstable regions in the  $(\rho, \mu)$  plane; for  $\sigma = 0$ , to the right of the broken lines; for  $\sigma = .1$ , to the right of the solid lines with a triple zig-zag. It will be noted that the single unstable range for  $\sigma = 0$  breaks up into several separate ranges for  $\sigma = 0.1$ , for small  $\rho$ , but continue as a single wider range for larger  $\rho$ . For large  $\sigma$ , the unstable speed ranges separate even further. The boundaries of the unstable region correspond to passage of one  $\lambda$ -pair from pure imaginary to real or complex values, and belong to the loci

$$\lambda_1^2 = 0, \quad \lambda_1^2 = -1, \quad \lambda_1^2 = \lambda_2^2 \quad (32)$$

corresponding to the roots of Equation 29. The geometry of the change is indicated in Figure 17.

$$x_1 = 0, \quad x_1 = -1, \quad x_1 = x_2$$

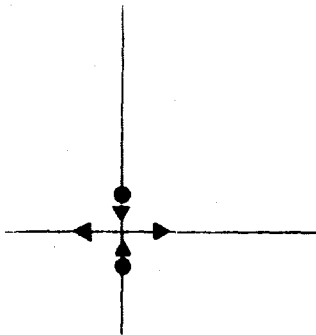


Fig. 17a

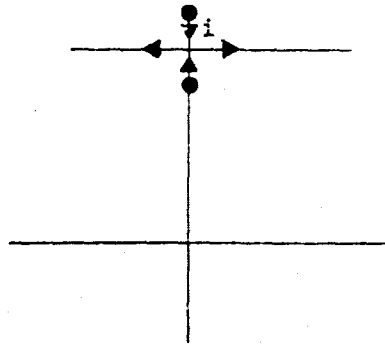


Fig. 17b

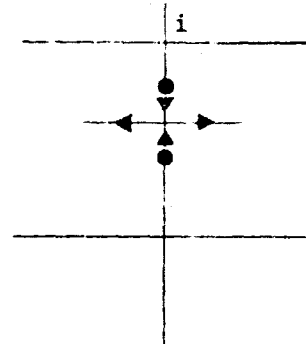


Fig. 17c

On Figure 16, the upper and lower curves correspond to  $\lambda_1^2 = 0$  and Figure 17a and "grow" out of the loci (18) for  $\rho = 0$ ; the middle one corresponds to  $\lambda_1^2 = \lambda_2^2$  and Figure 17c.

For small  $\rho, \sigma$  the above unstable ranges occur near  $\mu = 1$ . It can be shown that similar, but smaller, unstable ranges occur near

$$\mu = 2^2, 3^2, 4^2, \dots \quad (34)$$

that is, when the rotor speed is near a submultiple of  $\omega_0$ :

$$\omega/\omega_0 = 1/2, 1/3, \dots \quad (35)$$

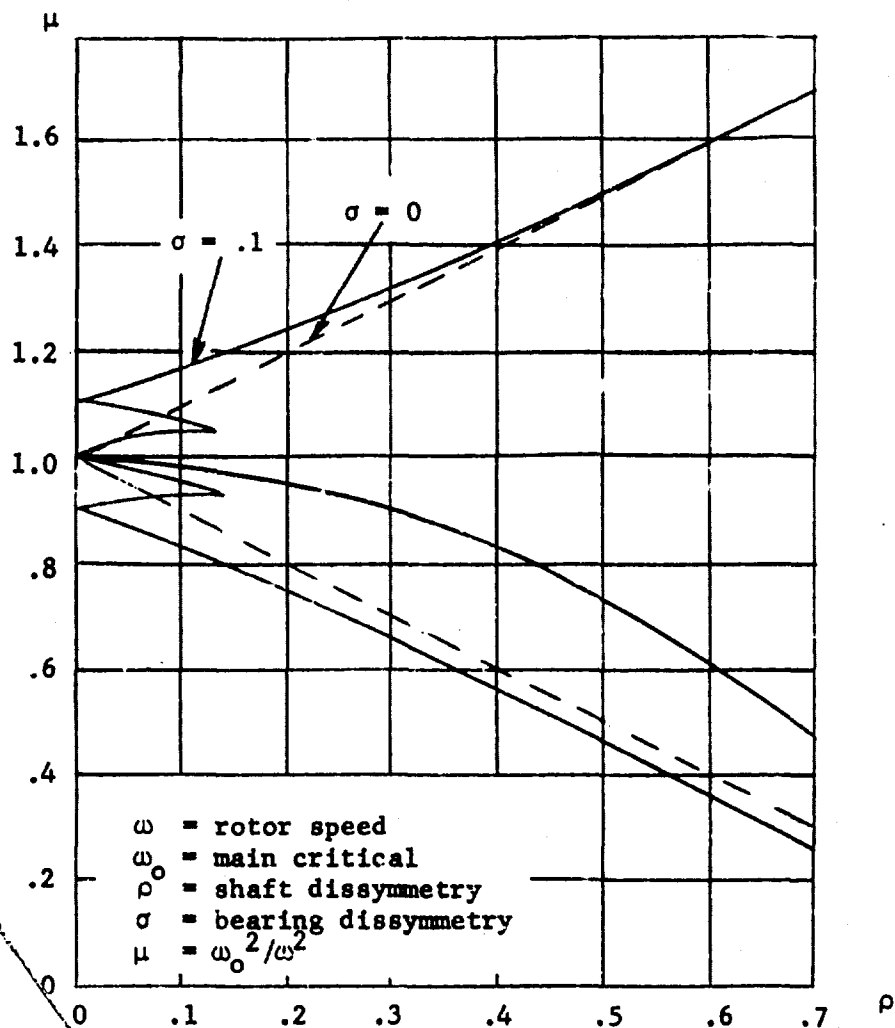


Fig. 16 Stable and Unstable Regions  
for  $\sigma = 0, .1$

The effect of static friction has been considered in Reference (10); it narrows the unstable range and may even abolish it.

Similar analysis is possible when unequal bearing support flexibilities are added on to the rotors of Section VIII, where both angular and translatory motion are included. In particular, the  $(\lambda, \mu)$  motion of the symmetrically mounted rotor of Section VI considered in the text near Equations 8(41), 8(42), with added  $f_1, f_2$  due to the bearing supports (converted to an equivalent angular stiffness constant), leads to a fourth order system with periodic coefficients. In form, these equations are the same as for a rigid shaft passing through the origin, with different angular stiffnesses for the angular deflections  $\lambda, \mu$ . This case has been studied by Brosen's and Crandall, Reference (12), using the same method of solution.

The addition of  $f_1, f_2$  to the general rotor of Section 8 leads to an eighth order system with periodic coefficients.

The method of convergence factors is applicable to higher order differential equations and will be discussed in another paper.

# XI

## APPLIED FORCES. GRAVITY. UNBALANCE

Thus far, the rotors have been assumed balanced, and free from external forces. We now consider briefly rotors subject to gravity, and unbalance forces and torques.

With the addition of external forces (and torques) the linear equations of motion become non-homogeneous. Their solution consists of the sum of a particular solution and a general solution of the homogeneous equations described in Sections II-VIII.

Particular solutions will now be considered corresponding to a constant force, such as gravity force; then to the unbalance of the rotor.

Under a constant applied force  $(X, 0)$  such as the gravity force  $\vec{X} = -mg$ , eqs. 2(1) become

$$m\ddot{x} + kx = X, \quad m\ddot{y} + ky = 0 \quad (1)$$

A particular solution is given by

$$x = X/k, \quad y = 0 \quad (2)$$

corresponding to the static shaft deflection under the action of this force. As stated above, to this is added the general solution of the homogeneous system 2(1). Hence, the motion can be described as the oscillation 2(2), not about  $x = 0, y = 0$ , but about the point (2).

Likewise, for eqs. 2(6), the solution consists of a sum of (2) and the damped oscillatory solutions of 2(6).

For the rotors of Sections III - V, a constant applied force produces a deflection in the direction of the force, as well as at right angles to it. In particular, under rotary friction, the addition of the force  $(X, 0)$  to the right-hand members of eqs. 4(1) yields the constant displacement given by

$$\begin{aligned} kx + f_1 \omega y &= X, \\ ky - f_1 \omega x &= 0, \end{aligned} \quad (3)$$

leading to

$$x = \frac{kX}{k^2 + f_1^2 \omega^2}, \quad y = \frac{f_1 \omega X}{k^2 + f_1^2 \omega^2}, \quad (4)$$

producing an angular displacement relative to the direction of the force, given by

$$\tan \alpha = \frac{f_1 \omega}{k}. \quad (5)$$

The above was utilized by A. L. Kimball in measuring the coefficient of internal damping, reference (13), by loading a horizontal rotating shaft vertically, and measuring its horizontal deflection. It had been expected that  $\alpha$  would vary with the rotor speed, and with the load; it turned out, however, that within elastic range of stress,  $\alpha$  was quite constant, and small. Thus, if a rotary friction model is used for internal damping, then  $f_1$  must be assumed to vary inversely as  $\omega$ .

Assuming the existence of a normal force  $\tilde{F}_n$  as in Section III, Eqs.3(2), but changing  $k_1$  to  $-k_1$ , one finds that a constant applied force  $(X, 0)$  produces for small  $k_1/k$  a displacement

$$x = \frac{X}{k}, \quad y = \frac{k_1 X}{k}, \quad \alpha = k_1/k. \quad (6)$$

The elastic energy of the bent shaft, due to the force  $X$  is given by

$$V = X^2/2k \quad (7)$$

Since the shaft is revolving, the elastic strain of any element of the shaft can be shown to vary sinusoidally with time. The existence of the force  $\tilde{F}_n$  can be explained by the assumption that the stress-strain behavior of the shaft material is described by narrow, elliptical hysteresis loops, and this assumption leads to the energy loss

$$\frac{\pi k_1}{k} V = \pi \alpha V \quad (8)$$

per cycle. Thus,  $\alpha$  is a measure of the "solid friction" or internal damping of the material.

For the rotor of Section 10, a constant applied force  $F_x$  produces a deflection given by eqs. 10( )

$$x = f_m F_x (1 + \sigma + \rho \cos 2\tau), \quad y = -f_m F_x \rho \sin x 2\tau \quad (9)$$

As  $\tau$  varies over half a rotation,  $\Delta\tau = \pi$ , this describes a circle, shown in Fig. 8, whose diameter is at

$$x = f_m F_x (1 + \sigma + \rho), \quad y = 0, \quad (10)$$

$$x = f_m F_x (1 + \sigma - \rho), \quad y = 0,$$

and whose radius is  $f_m F_x \rho$ . The motion along this circle is counter-clockwise.

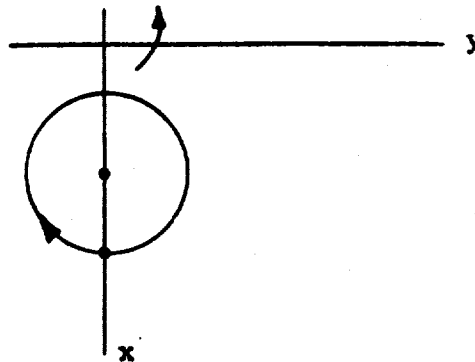


Fig. 18

We now consider the effect of unbalance. Assume the rotor symmetrically mounted between two bearings, as in Fig. 1, and suppose that the unbalance is likewise symmetrical, so that  $O_2$ , the c.m. of the rotor, lies in the midplane normal to the  $z$ -axis, at a distance  $\epsilon$  from  $O_1$ , the shaft center.

Denote by  $(x, y)$  the components of the displacement of  $O_1$ , from the unbent shaft center portion  $O$ . The position of the c.m.  $O_2$ , when the shaft is rotating, is

$$x + \epsilon \cos \omega t, \quad y + \epsilon \sin \omega t, \quad (11)$$

assuming that as  $t = 0$ ,  $O_1 O_2$  is parallel to the x-axis. The dynamical equations 2(1) become

$$m \frac{d^2}{dt^2} (x + \epsilon \cos \omega t) = -kx, \quad m \frac{d^2}{dt^2} (y + \epsilon \sin \omega t) = -ky \quad (12)$$

and introducing  $\omega_0$  as in 2(2),

$$\ddot{x} + \omega_0^2 x = \epsilon \omega^2 \cos \omega t, \quad \ddot{y} + \omega_0^2 y = \epsilon \omega^2 \sin \omega t. \quad (13)$$

The effect of the unbalance is thus the same as of an applied rotating force, equal to the centrifugal force.

A particular solution of eqs. (13) is

$$x = \frac{\epsilon \omega^2 \cos \omega t}{\omega_0^2 - \omega^2}, \quad y = \frac{\epsilon \omega^2 \sin \omega t}{\omega_0^2 - \omega^2} \quad (14)$$

Thus, the shaft center point  $O_1$  describes a circle of radius  $r$ , where

$$\frac{r}{\epsilon} = \left| \frac{\omega^2}{\omega_0^2 - \omega^2} \right| \quad (15)$$

and  $\overrightarrow{O O_1}$  is in the same direction as  $\overrightarrow{O_1 O_2}$  if

$$\omega^2 < \omega_0^2, \quad (16)$$

but in opposite directions if

$$\omega^2 > \omega_0^2 \quad (17)$$

Eqs. (14) imply that, for speeds below the critical, the rotor rotates with "the heavy side out"; at speeds above the critical, it moves with the "heavy side in". See Fig. 19.

A plot of  $r/\epsilon$  vs.  $\omega$  is shown in Fig. 20, Page 69.

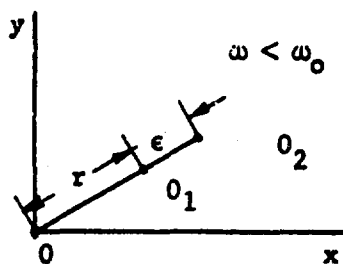


Fig. 19

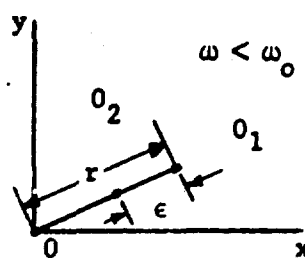


Fig. 20

where the ordinate plotted represents the right-hand member of (15), but without the absolute value sign.

From the above, it is evident that the critical speed  $\omega_0$  can be defined as the speed at which an unbalance causes a large (theoretically infinite) amplitude of shaft displacement. The angle between  $OO_1$  and  $O_1O_2$  changes from 0 to  $180^\circ$  at the critical.

As  $\omega$  becomes infinite, the ordinate of Fig. 19 approaches -1. This means that  $O_2$  approaches 0 and the rotor rotates about its c.m., which approaches the z-axis.

At  $\omega = \omega_0$ , a particular solution of (2) is given by

$$x = \frac{A \epsilon \omega_0 t}{2} \sin \omega_0 t, \quad y = -\frac{\epsilon \omega_0 t}{2} \cos \omega_0 t \quad (18)$$

This may be described as motion in a spiral, known as "arithmetic" or "Archimedean" spiral, with the radius  $r$  increasing linearly with time, indefinitely.

If a "static" friction force  $(-fx^0, -fy^0)$  is added to the right-hand member of (12), then, introducing  $Z = x + iy$  as in 3( ), the dynamical equation becomes

$$\ddot{Z} + \frac{f}{m} \dot{Z} + \omega_0^2 Z = \epsilon \omega^2 e^{i\omega t} \quad (19)$$

This leads to the particular solution

$$\frac{z}{\epsilon} = \frac{\omega^2 e^{i\omega t}}{-\omega^2 + \frac{if\omega}{m} + \omega_o^2} \quad (20)$$

whence, taking real and imaginary parts,

$$x = r \cos (\omega t - \phi), \quad y = r \sin (\omega t - \phi) \quad (21)$$

where (see Fig. 20)

$$\frac{r}{\epsilon} = \frac{\omega^2}{[(\omega^2 - \omega_o^2)^2 + f^2 \omega^2 / m^2]^{1/2}}, \quad (22)$$

$$\tan \phi = f\omega / m (\omega_o^2 - \omega^2). \quad (23)$$

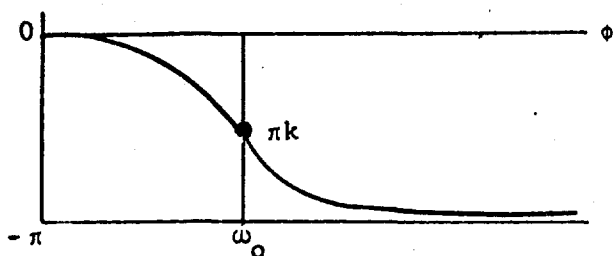


Fig. 21

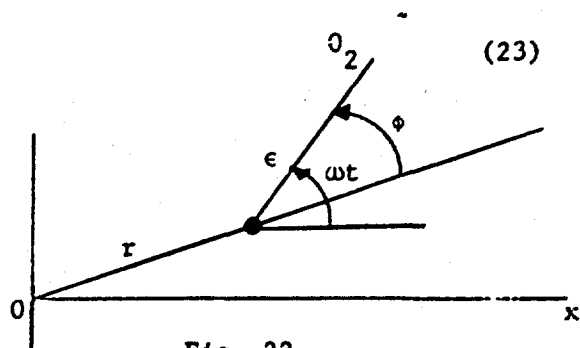


Fig. 22

A schematic plot of  $\phi$  vs.  $\omega$  is shown in Fig. 22, and  $r/\epsilon$  vs.  $\omega$  is in Fig. 23. There is now a gradual change in  $\phi$ , from 0 to  $-180^\circ$ , as  $\omega$  increases.

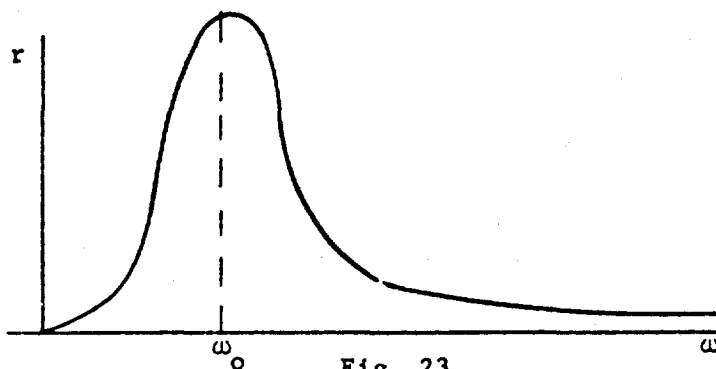


Fig. 23

It is of interest to compare (20) with

$$\frac{I}{E} = \frac{i\omega/L}{-\omega^2 + j \frac{R}{L} \omega + \omega_o^2}, \quad \omega_o^2 = \frac{1}{LC} \quad (24)$$

for the ratio of the current  $I = I_0 e^{i\omega t}$  to an applied voltage  $E = E_0 e^{i\omega t}$  in an R, L, C circuit. The typical resonance curve is obtained by plotting the absolute value of (24) vs.  $\omega$ . Essentially, (20) differs from (24) in having an extra factor  $i\omega$  in the numerator. The maximum of  $Z/\epsilon$  occurs at  $\omega = \omega_m > \omega_0$ , while for (24) it occurs at  $\omega = \omega_0$ . However, if  $f \omega_0/m$  is small, as is generally the case,  $\omega_m - \omega_0$  is also small.

Thus, unbalance causes a rotor whirl or vibration, whose amplitude is large near the speed  $\omega = \omega_0$ . This is the reason for the term "critical" for this speed, which was initially defined in 2(2) as the vibration frequency of the rotor mass  $m$  vs. the spring stiffness  $k$ .

For the rotors of Sections 3-5, a particular solution due to unbalance is still given by a circular motion of  $O_1$ , described by equations similar to (18)-(23), but with  $r/\epsilon$  and  $\phi$  given by different functions of  $\omega$ . This is the case, even when the solution of the homogeneous system is unstable, as for rotors with a normal force  $F_n$  as in eq. 3(1).

Addition of unequal bearing support flexibilities to the above rotors and shafts yields two criticals  $\omega_1, \omega_2$ . Now unbalance leads to an elliptic path for  $O$ , and to large amplitudes at each critical. The phase changes of  $\phi$  for each one near  $\omega_1, \omega_2$ , have the effect of producing for  $\omega_1 < \omega < \omega_2$  a retrograde motion of  $O_1$ , that is, opposite in direction to the rotation of the rotor.

Likewise, for the symmetrically mounted rotors of Section 10 with different shaft and different bearing flexibilities, unbalance leads to a periodic solution for the path of  $O$ . For small  $\rho$ , the path is nearly elliptical.

Thus far, only "translational unbalance" has been considered, involving  $(x, y)$  but not  $(\lambda, \mu)$ . We now turn to the rotors of Section VIII, under General unbalance, where the rotor displacement involves  $x, y$  and  $\lambda, \mu$ , and the unbalance involves both c.m. off the  $z$ -axis, and the principal axis of inertia off the  $z$ -axis.

Adopting the rotating axes of Section VIII, we suppose that the rotor, c.g., for the unbent shaft, is located off the  $z$ -axis, at

$$x = \epsilon_x, \quad y = \epsilon_y, \quad z = z_0 \quad (25)$$

We suppose further that a line through the c.m. parallel to the z-axis is not a principal axis of inertia of the (undeflected) rotor, through its c.m. Put Eq. 8(10) in the form

$$\omega = \omega_{x_2} i_2 + \omega_{y_2} j_2 + \omega_{z_2} k_2 = (\dot{\lambda} - \omega\mu) i_2 + (\dot{\mu} + \omega\lambda) j_2 + \omega k_2 \quad (26)$$

Then eq. 8(12) is replaced by

$$\begin{aligned} M_{\sim} = & i_2 \left[ I \omega_x - I_{xy} \omega_{y_2} - I_{yz} \omega_{z_2} \right] \\ & + j_2 \left[ -I_{xy} \omega_{x_2} + J \omega_{y_2} - I_{yz} \omega_{z_2} \right] \\ & + k_2 \left[ -I_{xz} \omega_{x_2} - I_{yz} \omega_{y_2} + K \omega_{z_2} \right] \end{aligned} \quad (27)$$

where  $I_{xy} \dots$  are the products of inertia about the  $x_2 y_2 \dots$  axes

If we assume "slight" rotary unbalances, then the products of inertia  $I_{xz}$ ,  $I_{yz}$  are small in comparison with the remaining moments and products of inertia.

Substituting for  $i_2$ ,  $j_2$ ,  $k_2$  from 8(9), differentiating with respect to time and recalling 8(3), one obtains  $i$ ,  $j$  - equations which differ from 8(15)

- (a) in the introduction of added terms linear in  $x$ ,  $y$ ,  $\lambda$ ,  $\mu$  on the left side of 8(15) involving  $I_{xy}$ ,  $I_{xz}$ ,  $I_{yz}$ , and
- (b) in acquiring the right-hand force and torque terms.

$$m\omega^2 \epsilon_x, \quad m\omega^2 \epsilon_y, \quad -I_{yz} \omega^2, \quad -I_{xz} \omega^2. \quad (28)$$

As regards the terms on the left which are linear in  $\lambda$ ,  $\mu$  and involve  $I_{xz}$ ,  $I_{yz}$ , these may be considered small of the second

order, and will be neglected. On the other hand, the terms involving  $I_{xy}$  are equivalent to the existence of an angle  $\gamma$  between the principal axes of the rotor and of the shaft cross-section. In absence of Fig. 28, one is led to a form of equations equivalent to those described at the end of Section VIII, involving the angle  $\gamma$ .

As regards the terms (28), the first two terms represent the unbalance forces corresponding to the right-hand members of (13); the last two, the unbalance moments due to the angle between the z-axis and the nearest principal axes of inertia.

Particular solutions of the resulting differential equations can be obtained by assuming that  $x, y, \lambda, \mu$  are constants, and are given by

$$x = \frac{N_x(\omega, \nu)}{\Delta(\omega, \nu)} \bigg|_{\nu=0} \dots \mu = \frac{N_\mu(\omega, \nu)}{\Delta(\omega, \nu)} \bigg|_{\nu=0} \quad (29)$$

where  $N_x \dots, N_\mu$  are obtained from  $\Delta$  by replacing a proper column of  $\Delta$  by the unbalance terms (28).

Since the axes of Section VIII are rotating axes, the solutions (29) represent a circular whirl of the rotor with the rotational speed  $\omega$ , with the unbalance forces and torques balanced by the elastic forces and moments. The amplitudes (29) become infinite as  $\omega$  approaches the "critical speeds" given by the  $\omega^2$  roots of  $\Delta(\omega, 0)$ . As pointed out in Section VIII, these critical speeds given by the  $\omega^2$  roots as transition points, at which the solutions of the homogeneous system 8(15) change from stable to unstable solutions. [Properly speaking,  $\Delta$  above represents not  $\Delta$  in Fig. 8(16), but obtained from the modified system resulting from the replacement of  $\underline{M}$  in Fig. 8(14) by Fig. (27).]

## XII

### VARIABLE SPEED

Thus far, we have always assumed that the rotational speed of the rotor,  $\omega$ , while arbitrary, is constant, without inquiring how this speed is maintained. Generally, the needed power for maintaining this speed is supplied by an electric motor, a turbine, a diesel, or some other "prime mover". It has thus been assumed that these "prime movers" supply whatever torque is needed to maintain the assumed constant rotational speed,  $\omega$ , even when the rotor is unstable and both kinetic and potential energy is required for the increasing amplitude of  $x$ ,  $y$ ,  $\lambda$ ,  $\mu$ .

In view of the large amplitude vibrations due to unbalance near the critical speed (See Equations XI (13), XI (19)), no rotor is ever designed to run at or near its critical speed if it has one such speed, or if it has several criticals, near any one of its criticals. If the operating speed of the rotor is above its critical(s), then it must pass through the critical(s). For a perfectly balanced rotor, as in Section II the rotation, and the translational motion,  $x, y$ , do not interfere with each other even when the rotor speed is variable, and the passage through the critical(s) is uneventful. On the other hand, for an unbalanced rotor whose speed is changing, the translational motion is affected by the variable speed. We proceed to consider this motion of an unbalanced rotor whose speed is uniformly accelerating from rest, so that its angle of rotation  $\theta$  and speed  $\dot{\theta}$  are given by

$$\theta = \alpha t^2/2, \quad \dot{\theta} = \alpha t. \quad (1)$$

Unless  $\alpha$  is very small, the solutions of Section XI say Equations XI (14), are quite inadequate for describing the rotor behavior.

Of special interest is the rotor motion near  $t = t_0$ , the time at which the rotor speed  $\dot{\theta} = \alpha t$  is equal to the critical speed  $\omega_0$ :

$$t_0 = \omega_0/\alpha. \quad (2)$$

For simplicity we consider the same rotor as in the first part of Section II, that is a centrally mounted rotor, with the c.m.  $O_2$  at a distance  $\epsilon$  from the shaft axis  $O_1$ , and with the principal inertia axis through  $O_2$  parallel to the  $z$ -axis. For such a rotor the translational motion  $x, y$  of the points  $O_1, O_2$  can be considered apart from the angular motion  $(\lambda, \mu)$  of the axis, and we proceed to consider the former.

Denote by  $x, y$  the position of  $O_1$ , the shaft center point in the plane containing  $O_2$ , the c.m. Then the position of  $O_2$ , assuming that  $\overrightarrow{O_1 O_2}$  is initially parallel to the  $x$ -axis, is given by

$$x' = x + \epsilon \cos \theta, \quad y' = y + \epsilon \sin \theta, \quad \theta = \alpha t^2 / 2.$$

In place of Equations 11(12) one now obtains the dynamical equations of motion:

$$\begin{aligned} \ddot{x} + \omega_0^2 x &= - \frac{d^2}{dt^2} (\epsilon \cos \frac{\alpha t^2}{2}), \\ \ddot{y} + \omega_0^2 y &= - \frac{d^2}{dt^2} (\epsilon \sin \frac{\alpha t^2}{2}). \end{aligned} \tag{4}$$

For definiteness we consider a particular solution of Equations (4) subject to the initial conditions of an unbent shaft, and starting from rest:

$$x = 0, \dot{x} = 0, y = 0, \dot{y} = 0 \quad \text{for } t = 0. \tag{5}$$

The general solution is obtained by adding the solution 2(3) to the above particular solution. (The effect of gravity changes the initial value of  $x$ .)

In terms of  $x', y'$  and the coordinates of the c.m., Equations (4) become

$$\begin{aligned} \ddot{x}' + \omega_0^2 x' &= \omega_0^2 \epsilon \cos(\alpha t^2 / 2) \\ \ddot{y}' + \omega_0^2 y' &= \omega_0^2 \epsilon \sin(\alpha t^2 / 2) \end{aligned} \tag{6}$$

while the boundary conditions (5) become

$$x' = \epsilon, \quad y' = 0, \quad \dot{x}' = 0, \quad \dot{y}' = 0;$$

these can be reduced to (5) by ——— the solution  $x' = \epsilon \cos \omega_0 t, y' = 0$ . Introducing the complex variable

$$Z = x' + iy' \tag{8}$$

one obtains from (6), (5)

$$\ddot{z} + \omega_0^2 z = \omega_0^2 e^{i\omega_0 t^2/2}, \quad z = 0, \dot{z} = 0 \text{ for } t = 0. \quad (9)$$

The solution of (9), or more generally, of

$$\ddot{u} + \omega_0^2 u = f(t), \quad u = 0, \dot{u} = 0 \text{ for } t = 0. \quad (10)$$

can be carried out in terms of its "Green's Function"  $G(t,s)$ , as follows

$$u(t) = \int_0^t G(t,s) f(s) ds, \quad (11)$$

where  $G$  is defined as follows:

$$\left\{ \begin{array}{l} G(t,s) = 0 \quad \text{for } t \leq s, \\ \frac{\partial^2 G}{\partial t^2} + \omega_0^2 G = 0 \quad \text{for } t > s \\ \left. \frac{\partial G(t,s)}{\partial t} \right|_{s=t} = 1. \end{array} \right. \quad (12)$$

Physically,  $G$  represents the solution of (10) for an "impulse" force at the time  $t = s$

$$f(t) = \delta(t-s) \quad (13)$$

where  $\delta$  is the "Dirac Function", while Equation (11) represents the response to any  $f(t)$  as a superposition of responses to equally spaced impulses.

The solution of (12) yields

$$G(t,s) = G(t-s) = \frac{\sin \omega_0(t-s)}{\omega_0} \quad \text{for } t > s.$$

The dependence of  $G$  on  $(t-s)$  is characteristic of linear differential equations with constant coefficients.

Applying (11), (14) to (9), we obtain

$$Z(t) = \omega_0 \epsilon \int_0^t \sin \omega_0 (t-s) e^{i\alpha s^2/2} ds \quad (15)$$

By expressing the sine in terms of imaginary exponentials, and completing the squares of the quadratic exponents, the (15) can be put in the form

$$A(t) = \frac{\omega_0 \epsilon}{2i} e^{-i\omega_0 t_0/2} \left[ e^{i\omega_0 t} \int_{-t_0}^{t-t_0} e^{i\alpha u^2/2} du - e^{-i\omega_0 t} \int_{t_0}^{t+t_0} e^{i\alpha u^2/2} du \right]. \quad (16)$$

The integral in (16) is related to the Fresnel integrals defined by

$$E(v) = C(v) + iS(v) = \int_0^v e^{i\pi\beta^2/2} d\beta,$$

$$C(v) = \int_0^v \cos(\pi\beta^2/2) d\beta,$$

$$S(v) = \int_0^v \sin(\pi\beta^2/2) d\beta.$$

There results

$$\left\{ \begin{aligned} Z(t) &= \frac{\omega_0 \epsilon}{2i} e^{-i\omega_0 t_0/2} \sqrt{\frac{\pi}{\alpha}} \left\{ e^{i\omega_0 t} [E(v-v_0) - E(-v_0)] - e^{-i\omega_0 t} [E(v+v_0) - E(v_0)] \right\} \\ \text{where} \\ v &= \frac{\alpha}{\pi} t, \quad v_0 = \frac{\alpha}{\pi} t_0 \end{aligned} \right\} \quad (18)$$

Tables of the functions  $C(v)$ ,  $S(v)$  may be found in Jahnke-Ende. The function  $E(v)$ , plotted in the complex  $O+iS$  plane, yields the well-known Cornu Spiral indicated schematically in Figure 24. For large  $v$ , the following asymptotic expression holds:

$$E(v) = \frac{1+i}{2} - \frac{i}{\pi} \frac{e^{i\pi v^2/2}}{v} + \dots \quad (19)$$

where .... refer to terms involving  $\frac{1}{3}$ ,  $\frac{1}{5}$ , ....

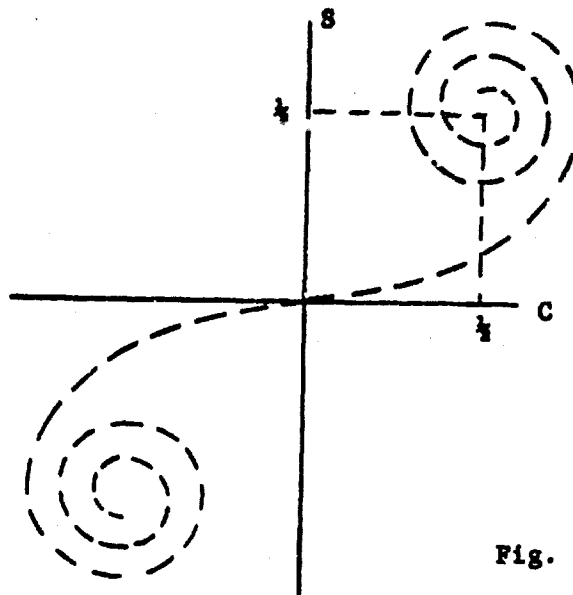


Fig. 24

The path described by  $O_2$  is given by

$$x' + iy' = Z(t) + \epsilon \cos \omega_0 t \quad (18)$$

and aside from the angle  $\theta = \alpha t^2/2$ , it depends on the angles

$$\theta_0 = \omega_0 t_0, \quad \theta(t_0) = \alpha t_0^2/2,$$

where  $\theta$  is the rotation of the shaft,  $\theta_0$  the rotation angle of a rotor rotating with constant angular velocity  $\omega_0$ . In particular, at the time  $t_0$ ,  $v = v_0$  and Equation (18) yields

$$Z(t_0) = \frac{\omega_0 t_0}{2} \sqrt{\frac{\pi}{\alpha}} \left\{ e^{i\theta_0/2} E(v_0) + e^{-3i\theta_0/2} [E(2v_0) - E(v_0)] \right\}. \quad (20)$$

The maximum of  $|Z|$  occurs generally for  $t > t_0$ .

A solution similar to (11) can be obtained for a rotor possessing static damping, as in Equations 2(6), provided  $G$  is replaced by

$$G = \frac{e^{-a(t-s)} \sin b(t-s)}{b}, \quad a + b = \lambda \quad (21)$$

where  $\lambda_1$  is given by II(9). Likewise, the integration can be expressed in terms of the integral  $E$  but for non-real arguments. The integral  $E$  is related to the error-function but again for complex arguments.

The assumption of Equation (1), even for a massive prime mover exerting a constant torque, is not necessarily valid, due to the variable torque that the shaft

must apply to maintain the motion of the unbalanced rotor. Indeed, the kinetic and potential energies of the rotor are given by

$$T = \frac{m}{2} \left\{ \left[ \frac{d}{dt} (x + \epsilon \cos \theta) \right]^2 + \left[ \frac{d}{dt} (y + \epsilon \sin \theta) \right]^2 \right\} + \frac{K}{2} \dot{\theta}^2$$

$$= \frac{m}{2} [\dot{x}^2 + \dot{y}^2 + 2\epsilon \dot{\theta} (\dot{y} \cos \theta - \dot{x} \sin \theta)] + \frac{K}{2} \dot{\theta}^2, \quad (22)$$

$$V = \frac{k}{2} (x^2 + y^2). \quad (23)$$

The first two Lagrange equations agree with (4) while the third one yields

$$\frac{d}{dt} [m\epsilon (\dot{y} \cos \theta - \dot{x} \sin \theta) + (m\epsilon^2 + K) \dot{\theta}] + m\epsilon \dot{\theta}^2 (\dot{y} \sin \theta + \dot{x} \cos \theta) = Q_{\theta}, \quad (24)$$

where  $Q_{\theta}$  is the external torque about the z-axis. If the above solution for  $x, y, \theta$  is valid, then the torque- $Q_{\theta}$  must act on the shaft. In general, the torsional stiffness coefficient of a shaft is of the same order of magnitude as its bending stiffness. Hence, the assumed Equation (1) must be corrected.

If the prime mover is not very massive, the effect of its moment of inertia and of its torque vs. speed characteristic must be taken into account.

### XIII

#### COMPOUND ROTORS

Thus far, in Sections II to XII, many basic features affecting rotor motion and stability have been considered. For clarity, the effect of each complicating feature was treated by itself, and under simplest possible assumptions. Thus the effect of bearing lubricant was considered in Section IX only for a symmetrically mounted rotor consisting of a single disk rotor, but it was neglected in Sections II to VIII, X to XII.

In practice, rotors may have more complex structures, and many of the features considered in Sections II to XII may be present simultaneously. As an example, a steam turbine rotor consists of a fairly massive shaft whose thickness varies axially, and on it are shrunk many disks carrying steam buckets. This "prime mover" may be coupled to a massive turbine generator. The "active length" of the latter forms a constant radius cylinder, carrying imbedded current coils in slots, and for a two-pole machine, may lead to different bending stiffnesses in two directions. The coupled, two-span rotor is mounted on several bearings and may have an overhung, small rotor (voltage regulator).

A complex rotor with distributed inertias and a variable thickness shaft is treated by breaking it up into many sections, each one with its own translatory and rotary inertias  $m_i$ ,  $I_i$ . The large number of degrees of freedom gives rise to a higher order system of differential equations, with more criticals. Inclusion of damping, bearings, pedestal support flexibility further complicates the system, very often to such an extent that high speed computing machines are required for their treatment.

To illustrate some of the features arising from an increase of degrees of freedom, we consider a rotor with two disks, as shown schematically in Figure 25.

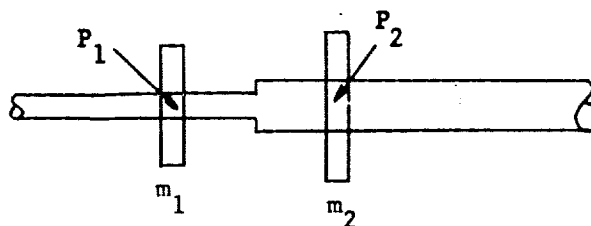


Fig. 25

We assume that the shaft and the disks are axially symmetric, and that each disk is perfectly balanced.

To set up the equations of motion, we start with the displacement load equations similar to those of Section VII.

$$\begin{aligned}
 x_1 &= a_{11}F_{x1} + a_{12}T_{y1} + a_{13}F_{x2} + a_{14}T_{y2}, \\
 \mu_1 &= a_{21}F_{x1} + a_{22}T_{y1} + a_{23}F_{x2} + a_{24}T_{y2}, \\
 x_2 &= a_{31}F_{x1} + a_{32}T_{y1} + a_{33}F_{x2} + a_{34}T_{y2}, \\
 \mu_2 &= a_{41}F_{x1} + a_{42}T_{y1} + a_{43}F_{x2} + a_{44}T_{y2}.
 \end{aligned}
 \tag{1}$$

Here  $x_1, \mu_1$  are the linear and angular deflections in the  $x, z$  plane of the shaft at  $P_1$ , the center point  $m_1$ ;  $x_2, \mu_2$  the corresponding deflections of  $P_2$ , the center point of  $m_2$ , while  $F_{x1}$  and  $T_{y1}$  are the external force in the  $x$ -direction, external torque about a line parallel to the  $y$ -axis, applied at  $P_1$ ;  $F_{x2}, T_{y2}$  a similar force and torque applied at  $P_2$ . Thus, if only a force  $F_{x1}$  is acting, then the deflections produced by it at  $P_1, P_2$  are given by the terms in the first column on the right of Equation (1) and so forth.

Solution of equations (1) yields

$$\begin{aligned}
 E_{x1} &= A_{11}x_1 + A_{12}\mu_1 + A_{13}x_2 + A_{14}\mu_2 \\
 T_{y1} &= A_{21}x_1 + A_{22}\mu_1 + A_{23}x_2 + A_{24}\mu_2 \\
 F_{x2} &= A_{31}x_1 + A_{32}\mu_1 + A_{33}x_2 + A_{34}\mu_2 \\
 T_{y2} &= A_{41}x_1 + A_{42}\mu_1 + A_{43}x_2 + A_{44}\mu_2
 \end{aligned}
 \tag{2}$$

where  $A_{ij}$  is the "stiffness matrix". Both the  $A_{ij}$  and the flexibility matrix  $a_{ij}$ , by Maxwell's reciprocity theorem, are symmetric:

$$a_{ij} = a_{ji}, \quad A_{ij} = A_{ji}
 \tag{3}$$

For an axially symmetric rotor, the  $(y, \lambda)$  equations are similar to Equations 1 and 2, and can be obtained from them by carrying out the following substitutions:

$$\begin{aligned} y_1 &\rightarrow x_1, \quad \mu_1 \rightarrow -\lambda_1, \quad y_2 \rightarrow x_2, \quad \mu_2 \rightarrow -\lambda_2, \\ F_{x1} &\rightarrow F_{y2}, \quad T_{y1} \rightarrow -T_{x2}, \quad F_{x2} \rightarrow F_{y2}, \quad T_{y2} \rightarrow -T_{y2} \end{aligned} \quad (4)$$

Consider first  $\omega = 0$ , that is, a non-rotating rotor. The dynamic equations of vibration in the  $(x, z)$ -plane are obtained from Equations (2) by replacing its left-hand members by the inertia forces and torques of the disks:

$$F_{x1} = -m_1 \ddot{x}_1, \quad T_{y1} = -I_1 \ddot{\mu}_1, \quad F_{x2} = -m_2 \ddot{x}_2, \quad T_{y2} = -I_2 \ddot{\mu}_2 \quad (5)$$

There results an eighth-order system of four differential equations. The vibrations in the  $(y, z)$ -plane follow under the substitutions (4)

Assuming a solution of either set of equations in which time enters as a factor,

$$e^{ipt}, \quad (6)$$

one is led to the following equation for  $p$ :

$$\Delta = \begin{vmatrix} y_1 & \mu_1 & y_2 & \mu_2 \\ x_1 & -\lambda_1 & x_2 & -\lambda_2 \\ A_{11} - m_1 p^2 & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} - I_1 p^2 & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} - m_2 p^2 & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} - I_2 p^2 \end{vmatrix} = 0 \quad (7)$$

where the first row on the top indicates the variable occurring in each column.

The  $(y, \mu)$  frequency equation is exactly the same, provided the variables be chosen in accordance with the substitution in Equation 4, hence as indicated by the second row above the determinant.

Consider next  $\omega > 0$ , that is a rotating rotor. For it, the rotary inertia terms must be corrected by the addition of gyroscopic terms. Recalling Equations 7(11) and 7(12), one obtains for  $\dot{\tilde{M}}_1, \dot{\tilde{M}}_2$ , the rate of change of moment of momentum of each disk,

$$\begin{aligned}\dot{\tilde{M}}_1 &= (I_1 \ddot{\lambda}_1 + K_1 \dot{\mu}_1 \omega) \tilde{i} + (I_1 \ddot{\mu}_1 - K_1 \dot{\lambda}_1 \omega) \tilde{j} \\ \dot{\tilde{M}}_2 &= (I_2 \ddot{\lambda}_2 + K_2 \dot{\mu}_2 \omega) \tilde{i} + (I_2 \ddot{\mu}_2 - K_2 \dot{\lambda}_2 \omega) \tilde{j},\end{aligned}\quad (8)$$

and the  $\tilde{i}$ -components of  $\dot{\tilde{M}}_1, \dot{\tilde{M}}_2$  are equated to  $T_{y1}, T_{y2}$ ; the  $\tilde{j}$ -components to  $-T_{x1}, -T_{x2}$ . There results now a system of eight differential equations of net order 16, involving all eight variables  $x_1, \dots, \mu_2$ .

It is possible to halve the order of the system by introducing the variables

$$Z_1 = x_1 + iy_1, \quad M_1 = \mu_1 - i\lambda_1, \quad Z_2 = x_2 + iy_2, \quad M_2 = \mu_2 - i\lambda_2. \quad (9)$$

By proper manipulation of the  $(x_1, \dots, \mu_2)$ -equations, there results a system similar to the  $(x, \mu)$ -equations, except for the addition, in the principal diagonal, of the terms

$$0, \quad -kK_1 \omega \dot{M}_1, \quad 0, \quad -iK_2 \omega \dot{M}_2. \quad (10)$$

Under the assumption in Equation (6), these added terms become

$$0, \quad K_1 \omega p M_1, \quad 0, \quad K_2 \omega p M_2, \quad (11)$$

and lead to the net inertia terms

$$-m_1 p^2 Z_1, \quad (-I_1 p^2 + K_1 \omega p) M_1, \quad -m_2 p^2 Z_2, \quad (-I_2 p^2 + K_2 \omega p) M_2.$$

There result in the variables of Equations (9) four simultaneous equations which are the same as the  $(y, \mu)$ -equations. The determinant of the coefficients, when equated to zero, resembles Equation 7 for a non-rotating rotor, provided  $I_1, I_2$  are replaced by

$$I_1' = I_1 - K_1 \frac{\omega}{p}, \quad I_2' = I_2 - K_2 \frac{\omega}{p}.$$

In general, for only  $\omega$ , the  $p$ -roots are real, and the solution represents a possible circular precession. If the critical speeds are now defined, as in Section 12, as the  $\omega$ -values for which the precessional frequency is equal to the rotational speed  $\omega$ :

$$p = \omega; \tag{14}$$

then Equations (13) are replaced by

$$I_1' = I_1 - K_1, \quad I_2' = I_2 - K_2. \tag{15}$$

Hence, we conclude that so far as the gyroscopic effect on the critical speeds is concerned, the static vibrational equations, say in the  $(x,z)$ -plane, can be used, provided the rotary inertias of each disk be modified as in Equation (15).

This rule applies to an axially symmetric rotor with any number of disks.

In practice even the calculation of the stiffness and flexibility matrices  $a_{ij}$ ,  $A_{ij}$  becomes irksome for a complex rotor. One proceeds by step-wise integration of the "beam equations", say for the deflections in the  $(x,z)$ -plane for  $x$  and  $\mu$ :

$$\mu = \frac{\partial x}{\partial z}, \quad EI \frac{d\mu}{dz} = -T_y, \quad \frac{dT_y}{dz} + S_x + m_y = 0, \tag{16}$$

$$\frac{dS_x}{dz} + w_x = 0$$

for the deflection  $x$ , slope  $\mu$ , moment  $T_y$ , shear  $S_x$ , where  $w_x$  is the external load per unit  $z$ ,  $m_y$  the external applied moment per unit  $z$ . Both  $w_x$  and  $m_y$  are derived from the inertia load of each section, and are concentrated, say by being divided equally at the two end points.

Assuming a value of  $\omega$ , one solves the fourth-order system, subject to proper boundary conditions at each end. For a single rotor span of length  $l$ , mounted on "fixed"

bearings at each rod, these conditions are

$$x = 0, \quad T_y = 0 \quad \text{at } z = 0,$$

(17)

$$x = 0, \quad T_y = 0 \quad \text{at } z = l.$$

One proceeds by applying Equation (17) and assuming several sets of further conditions at  $x = 0$ , say the values of  $\mu$  and  $S_f$ . The resulting solutions are then interpolated to satisfy, say  $x = 0$  at  $z = l$ , and  $T_y$  at  $z = l$ , is computed. Then  $\omega$  is varied until  $T_y$  vanishes at  $z = l$ . See References (16), (17).

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## 13. ABSTRACT

This report treats the theory of the stability of an elastic rotor. It covers such effects as instability caused by friction damping in the rotor, instability caused by shaft asymmetry, instability caused by asymmetry of the rotor mass, instability induced by fluid film journal bearings, the effect of static damping on the stability of the rotor, the influence of gyroscopic effects, the effect of flexible bearing supports, and the effect of gravitational and unbalance forces. The report gives the basic governing equations and the methods of their solution. The conditions are established under which instability is encountered and it is shown how the results relate to a practical rotor system.

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